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KINEMATICAL APPROACH FOR CONTACT PROBLEMS WITH ARBITRARY LARGE DEFORMATIONS

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A kinematical approach, based on the consideration of contact from the surface geometry point of view, is used for a consistent formulation of contact conditions and for the derivation of the corresponding tangent matrix. Within this approach differential operations are treated as covariant derivatives in the local surface coordinate system. The main advantage is a more algorithmic and geometrical structure of the tangent matrix, which consists of a ñmainò, a ñrotationalò and a pure ñcurvatureò term. Representative examples with contact and bending of shells modeled with linear and quadratic elements over some classical second order geometrical figures serve to show situations where keeping all parts of the tangent matrix is not necessary.

1. Introduction. From the variety of methods, which are mainly used for the solution of contact problems, the ñmaster-slaveò concept is one of the most robust methods. This concept is based on the determination of the penetration of the ñslaveò surface, represented by ñslaveò nodes into a ñmasterò surface. The penetration can be used for regularization methods like the penalty method and the Augmented Lagrange multiplier method. The penalty method, see e.g. Wriggers and Simo [1], Laursen and Simo [2], leads to the exact solution in the limit when the penalty approaches infinity. In nonlinear contact problems the penetration is a function of the current geometry and it is used for the ñconstitutiveò model of the contact forces. For the solution of nonlinear equilibrium equations by a Newton method the corresponding equations have to be linearized. The main idea of the proposed kinematical approach is to consider the global linearization separately from the local ñslaveò node searching procedure and derive linearized equations from kinematic equations in the local surface coordinate system. It leads to a very simple structure of the tangent matrix for the contact element, which is naturally divided into a ñmainò, a ñrotationalò and a ñpure curvatureò parts. For extensive tests of the proposed technique numerical examples with curved surfaces are performed. These tests serve to check the influence of different parts of the contact matrix on convergence within a nonlinear solution process.

2. Contravariant formulation of contact conditions. We introduce two coordinate systems: a reference global coordinate system for the finite element discretization only and a spatial local surface coordinate system in the contact consideration. All geometric properties of the element as well as the differential operations will be described in the local coordinate system of the ñmasterò surface. A surface of this element is parameterized by local coordinates ξ^1, ξ^2 from the finite element discretization. We introduce surface coordinate vectors in the usual fashion:

$$r_i = \frac{\partial r_i}{\partial \xi^i}\,,\ i = 1{,}2\,,\ n = \frac{r_1 \times r_2}{\left|r_1 \times r_2\right|}\,. \tag{1}$$

As a necessary procedure to define the value of penetration, see in Wriggers [3] and Laursen [4], the projection procedure of the contact node vector onto the master surface has to be provided. Let r_s be a position vector of a ñslaveò vector and r its projection onto the ñmasterò surface. Then the standard closest point procedure to define the value of the penetration is written as the following extremal problem:

$$(r_s - r)\cdot(r_s - r) \to \min\,. \tag{2}$$

This problem can be solved e. g. by the Newton method.

As the next step we construct a special local coordinate system, introducing the third coordinate ξ^3 in the direction of the surface normal n, and keeping a surface point $r(\xi^1, \xi^2)$ as a projection of the ñslaveò point:

$$r_s\left(\xi^1, \xi^2, \xi^3\right) = r\left(\xi^1, \xi^2\right) + \xi^3 n\,. \tag{3}$$

One should notice, that the projection procedure is taken into account within our local coordinate system. The Lie type derivative in the form of a covariant derivative [5] is used for any differential operation on the surface. For this we now consider the motion of the ñslaveò point in the local coordinate system, assuming that the ñmasterò surface is moving. Within a static process, the time t is treated as an incremental load parameter. Then the full time derivative of the ñslaveò point after taking into account Weingarten formula, known from differential geometry, and after some algebra transformation leads to the following expression:

$$v_s = v + \xi^3 \frac{\partial n}{\partial t} + n\dot{\xi}^3 + \left(r_j - \xi^3 h_j^i r_i\right)\dot{\xi}^j\,,\ i, j = 1{,}2\,, \tag{4}$$

where $v_s = \dfrac{\partial r_s}{\partial t}$ is velocity of the ñslaveò point and v is velocity of its projection onto the ñmasterò surface respectively, and h_j^i are mixed components of the curvature tensor for the ñmasterò surface.

All further considerations are based on the following assumption: only the contact problem is considered, but not the motion and deformation of the two body system connected by means of the normal vector with coordinate ξ^3; the penetration ξ^3 is assumed to be very small, as usual during the solution of contact problems.

The global iteration procedure for the solution leads to a decreasing value of the penetration g. Thus, eq. (4) for the convective velocity, with the additional assumption $\xi^3 = 0$ can be simplified to the form:

$$\dot{\xi}^{j} = a^{ij}(v_s - v)\cdot r_i,\ i, j = 1,2, \tag{5}$$

For the time derivative of the third coordinate ξ^3, after taking a dot product with the normal n and eq. (4) we have:

$$\dot{\xi}^{3} = (v_s - v)\cdot n,\ i, j = 1,2. \tag{5}$$

3. Variational equation for the contact. We define contact tractions T_c, T on the master surface s_c as well as on the slave surface s in the current configuration respectively. On each surface the contact traction is split into the normal and tangential traction, e. g. for the master surface it has the form:

$$T = Nn + \xi^i r_i. \tag{6}$$

Then the weak equilibrium equation, having taken into account eq. (4) in the sense of variations, has the form:

$$\delta W_c = \int_s N\delta\xi^3 ds + \int_s \left[a_{ij}T^i\delta\xi^j ds + \xi^3 T^i\left(\delta n\cdot r_i - h_j^k a_{ik}\delta\xi^j\right)\right]ds. \tag{7}$$

One can show, that the main part of the contact integral (7) after consistent expansion into a Taylor series with the small parameter ξ^3 with taking into account the expansion for the convective velocity (5) has the following form:

$$\delta W_c = \int_s N\delta\xi^3 ds + \int_s a_{ij}T^i\delta\xi^j ds. \tag{8}$$

Therefore, we have the description of the contact conditions of the first order with respect to the value of penetration $g = \xi^3$.

4. Penalty regularization of the contact conditions. The condition of non-penetration into the ñmasterò surface can be satisfied within the penalty method, which leads to the following functional:

$$\delta W_c = \int_s \varepsilon_N \langle g\rangle \delta g ds. \tag{9}$$

where ε_N is a penalty parameter, and $\langle\bullet\rangle$ is a Macauley bracket defined as follows:

$$\langle g\rangle = \begin{cases} 0, \text{if } g > 0 \\ g, \text{if } g \le 0 \end{cases}, \tag{10}$$

which means that the contact (9) is taking into account only in the case of non-positive value of the penetration g.

5. Linearization of the variation contact equation. The whole contact problem is nonlinear and can be solved by some of the iterative method, e.g. by Newton method. For this we have to calculate the constitutive derivative of the nonlinear system. Besides the global equilibrium equation we have to define the linearized equation of the variational contact integral (10). The fully linearized contact integral (9) has the following form:

$$D\delta W_c =$$

$$= \int_s \varepsilon_N H(-g)(\delta r_s - \delta r)\cdot(n\otimes n)(v_s - v)ds - \tag{11}$$

$$-\int_s \varepsilon_N H(-g) g\left(\delta r_{,j}\cdot a^{ij}(n\otimes r_i)(v_s - v) + (\delta r_s - \delta r)\cdot a^{ij}(r_j \otimes n)v_{,i}\right)ds - \tag{12}$$

$$-\int_s \varepsilon_N H(-g) g(\delta r_s - \delta r)\cdot h^{ij}(r_i\otimes r_j)(v_s - v)ds\,, \tag{13}$$

where $H(-g)$ is the Heaviside function, replacing the Macauley brackets. The full contact tangent matrix is then directly subdivided into the ñmainò part (11), the ñrotationalò part (12) and the ñcurvatureò part (13).

6. Finite element discretization. We consider details of the finite element implementation for in the case of ñnode-to-surfaceò approach. A contact element inherits the geometry from the finite element mesh of a contacting body, therefore, a nodal displacement vector is defined in the usual fashion:

$$u^T = \left\{u_1^{(1)}, u_2^{(1)} u_3^{(1)}, u_1^{(2)}, u_2^{(2)} u_3^{(2)}, \ldots, u_1^{(n)}, u_2^{(n)} u_3^{(n)}, u_1^{(n+1)}, u_2^{(n+1)} u_3^{(n+1)}\right\}, \tag{14}$$

where the first n nodes belong to the master surface, while the (n+1)ôth node is the ñslaveò node. Introduce a position matrix A and a matrix of the shape function derivatives $A_{,i}$:

$$A = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \ldots & N_n & 0 & 0 & 1 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \ldots & 0 & N_n & 0 & 0 & 1 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \ldots & 0 & 0 & N_n & 0 & 0 & 1 \end{bmatrix}; \tag{15}$$

$$A_{,i} = \begin{bmatrix} N_{1,i} & 0 & 0 & N_{2,i} & 0 & 0 & \ldots & N_{n,i} & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{1,i} & 0 & 0 & N_{2,i} & 0 & \ldots & 0 & N_{n,i} & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{1,i} & 0 & 0 & N_{2,i} & \ldots & 0 & 0 & N_{n,i} & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

After transformation of (11), (12), (13) we have the following part of the tangent matrix:

the main part

$$K^{(m)} = \varepsilon_N \int_s H(-g) A^T (n\otimes n) A ds\,; \tag{17}$$

the rotational part

$$K^{(r)} = -\varepsilon_N \int_s H(-g) g\left(A^T{}_{,j}\cdot a^{ij}(n\otimes r_i)A + A^T(\delta r_s - \delta r)\cdot a^{ij}(r_j\otimes n)A_{,i}\right)ds\,; \tag{18}$$

the curvature part

$$K^{(c)} = -\varepsilon_N \int_s H(-g) g A^T h^{ij}(r_i \otimes r_j) A ds\,. \tag{19}$$

The full normal tangent matrix is a sum of three parts:

$$K = K^{(m)} + K^{(r)} + K^{(c)}. \tag{20}$$

7. Numerical examples. The proposed procedure has been implemented into the finite element code FEAP-MeKA documented in [6]. Various shell structures were modeled with the family of ñsolid-shellsò elements [7], [8]. The series of numerical examples serves to investigate the influence of different parts of the tangent matrix on the convergence of the iterative algorithm. Contact problems between a flexible structure and rigid surfaces of second order (cylinder and sphere) are investigated. Flexible structures were modeled as elastic shells satisfying the St. Venant material law. In order to investigate the corresponding contribution of each part of the tangent matrix, the following three alternatives are considered: **a)** use of the full tangent matrix; **b)** use the main part and the rotational part; **c)** use the main part and the pure curvature part; **d)** use only the main part. The number of equilibrium iterations at each load step gives the influence of each case on the convergence rate. Table 1 shows the number of equilibrium iteration (No) for the number of the load step (No l.s.) and the cumulative number of iteration (Cum. No) in the cases **a, b, d** mentioned above. The contribution of each part in eq. (20) is measured by the norm $\varepsilon = \frac{\left\| K - K^{(resp)} \right\|}{\left\| K \right\|} \cdot 100\%$ and presented in table 1 as well. Fig. 1 shows both the initial and the final state for bending a shell over a rigid cylinder and fig. 2 shows the case with bending a shell over a rigid sphere. A shell was clamped in both examples and loaded incrementally by prescribed displacements in the first example and by nonsymmetrical forces in the second example. The shell surface was represented as a ñmasterò surface during computation.

Figure 1. *Initial and final state for bending a beam over a rigid cylinder.*

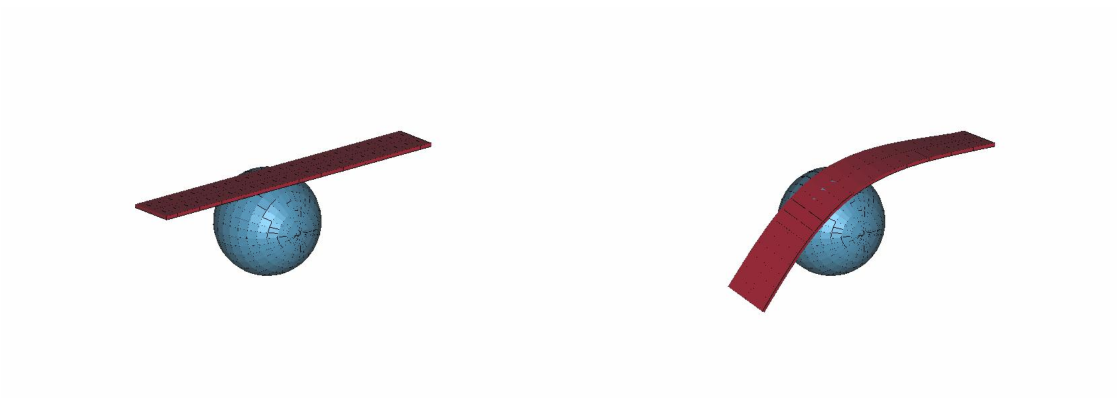

Figure 2. *Initial and final state for bending a shell over a rigid sphere.*

8. Conclusions. The proposed kinematical approach for the development of a consistent contact tangent matrix allows to distinguish between three parts of a tangent matrix, namely the ñmainò part, the ñrotationalò part and the ñpure curvatureò part. The numerical examples show that in the case of linear approximations and aligned contact elements keeping of the ñpure curvatureò part is meaningless. If elements with higher order approximations are used, the influence of the ñrotationalò part is larger, but the influence of the ñpure curvatureò part remains still small. Therefore, the last part, which is computationally more expensive then the others, can be eliminated from the complete tangent matrix without loss of efficiency.

Case a.				**Case b.**			**Case d.**		
No l.s.	No.	Cum. No.	$\varepsilon \cdot 10^{-2}$, %	No l.s.	No.	Cum. No.	No l.s.	No.	Cum. No.
1	27	27	0.731	1	27	27	1	20	20
2-23	4	115	7.382	2-22	4	111	2-8	4	48
24	6	121	18.08	23	6	117	9-23	5	123
25-33	4	157	16.26	24-33	4	157	24	6	129
34	5	162	12.20	34	6	163	25-33	5	174
35-100	4	426	17.32	35-100	4	427	34-48	6	264
							49-58	5	314
							59-68	6	374
							69-85	7	493
							86-100	8	613

Table 1. *Bending over a rigid sphere. Biquadratic element. Influence of various parts of the tangent matrix on convergence.*

References

1. Wriggers P., Simo J.C./A note on tangent stiffness matrices for fully nonlinear contact problems //Communication in Applied Numerical Methods, 1985, 1, pp. 199-203.
2. Laursen T.A., Simo J.C./A continuum-based finite element formulation for the implicit solution of multibody large deformation frictional contact problems//International Journal for Numerical Methods in Engineering, 1993, 35, pp. 3451-3485.
3. Wriggers P. Computational Contact Mechanics. John Wiley & Sons, 2002, 464 P.
4. Laursen T.A. Computational Contact and Impact Mechanics. Fundamentals of Modeling Interfacial Phenomena in Nonlinear Finite Element Analysis, Springer, Berlin Heidelberg New York, 2002, 454 P.
5. Marsden, J.E., Hughes, T.J.R. Mathematical foundations of elasticity. Prentice-Hall, Inc., 1983, 556 P.
6. Taylor R.L. FEAP - A Finite Element Analysis Program. University of California at Berkeley, Berkeley. Schweizerhof, K. and Coworkers, FEAP - Mechanik - Karlsruhe, Institut für Mechanik, Universität Karlsruhe.
7. Hauptmann R., Schweizerhof K. /A systematic development of ñsolid-shellò element formulation for linear and non-linear analysis employing only displacement degrees of freedom//International Journal for Numerical Methods in Engineering, 1998, 42, pp. 49-69.
8. Hauptmann R., Schweizerhof K., Doll S. /Extension of the ñsolid-shellò concept for applications to large elastic and large elastoplastic deformations//International Journal for Numerical Methods in Engineering, 2000, 49, pp. 1121-1141.