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ON AN EFFICIENT MODEL FOR GAS AND FLUID SUPPORTED MEMBRANES AND SHELL STRUCTURES

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Abstract. *The large deformation analysis of membrane or shell structures loaded and/or supported by gas or fluid can be based on a finite element description for the structure only. Then in statics the effects in the gas or the fluid have to be considered by using the equations of state for the gas/fluid, the information about the current volume and the current shape of the structure. The interaction with the structure is then modeled by a pressure resulting from the gas/fluid always acting normal to the current wetted structural part. This description can be easily used to model the filling process without all the difficulties involved with standard discretization procedures. In addition the consistent derivation of the nonlinear formulation and the linearization for a Newton type scheme results in a particular formulation which can be cast into a very efficient solution procedure based on a sequential application of the Sherman-Morrison formula.*

1 INTRODUCTION

The simulation of the inflation and the support of thin membrane or shell type structures can be usually performed in an efficient way by assuming an internal pressure in the structure which acts normal to the inner surface [2], [3], [11]. The restriction of this model is that it does not take into account the change of the inner volume of the structure due to the deformation of the structure even if there is no further inflation. Also the pressure may change due to temperature modifications. In both cases the volume of the gas has to be considered in the model [1], [8]; the latter is also important for external loading and associated stability considerations of the gas filled structure. Then in the case of gas filling the internal gas pressure formally provides an additional rank-one update of the FE stiffness matrix which stabilizes from an engineering point of view an almost completely flexible structure. In the case of fluid filling [5], [6], [9] or a mixture of gas and fluid [7], the filling of membrane-like structures can be performed without a separate discretization of the fluid with e.g. FE or Finite Volumes or similar. Also fully fluid filled structures can be analyzed without separate discretization of the fluid [7]. Several cases have to be distinguished, fluid with open surface [6], [9], structure under overpressure of fluid [7] and fluid with free surface but gas overpressure. The contribution shows the derivation of the variational formulation and the corresponding Finite Element discretization for compressible fluids under gravity loading. A particular focus is on the consistent linearization of the nonlinear equations and the accompanying constraint equations. Also the specific solution of the linearized equation system based on a sequential application of the Sherman-Morrison formula is presented. As a numerical example the large deformation analysis of a fully fluid filled shell structure with thin flexible walls is presented.

2 GOVERNING EQUATIONS

The mathematical description of static fluid structure interaction can be based on the principle of stationarity for the total potential energy δW of a fluid in an elastic structure and additional equations describing the physical behavior of different fluids or gases.

2.1 Virtual work expression

The variation of the elastic potential of the structure is specified by $\delta^{el}V$, $\delta^{i}\Pi$ denotes the virtual work of the pressure loading which acts between the fluid i and the structure, $\delta^{ex}\Pi$ is the virtual work of other external forces acting on the structure.

$$\delta W \quad = \quad \delta^{el}V + \delta^{i}\Pi - \delta^{ex}\Pi = 0 \tag{1}$$

The interaction term between liquid and structure is described by a body fixed pressure force ${}^{i}p\ {}^{*}\mathbf{n}$, with a non-normalized normal vector ${}^{*}\mathbf{n} = \frac{\mathbf{e}_\xi \times \mathbf{e}_\eta}{|\mathbf{e}_\xi \times \mathbf{e}_\eta|}$ and the pressure level ${}^{i}p$, see equation (2). $\mathbf{e}_\xi$, $\mathbf{e}_\eta$ denote covariant unit vectors on the wetted surface of the structure.

$$\delta^{i}\Pi \quad = \quad \int_\eta \int_\xi {}^{i}p\ {}^{*}\mathbf{n} \cdot \delta\mathbf{u}\ d\xi d\eta \tag{2}$$

The pressure acts normal to the surface element $d\xi d\eta$ along the virtual displacement $\delta\mathbf{u}$. Therefore a virtual work expression of a follower force is given. Possible physical properties of the fluid i are summarized in the following paragraphs:

2.2 Compressible fluids

If the dead weight of a fluid is neglected, we can distinguish between a pneumatic model, see [1], [8] and a hydraulic description. The corresponding constitutive equations are the Poisson's law for a pneumatic ($i = p$) and the Hooke's law ($i = h$) for a hydraulic model.

2.2.1 Pneumatic model

In realistic physical situations the investigations can be restricted to conservative models, which entails the application of the adiabatic state equation.

$$ {}^{p}p\; v^{\kappa} \;=\; {}^{p}P\; V^{\kappa} = const. \tag{3} $$

${}^{p}p,\ v$ are the state variables (pressure and volume) of the gas in the deformed state, capital letters denote the initial state and κ the isentropy constant.

2.2.2 Hydraulic model

For an analysis of hydraulic systems the liquid pressure is given by Hooke's law. ${}^{h}p$ is the mean pressure in the liquid determined by the bulk modulus K and the relative volume change of the liquid.

$$ {}^{h}p(v) \;=\; -K\frac{v-V}{V} \tag{4} $$

2.3 Hydrostatic loading - Incompressible fluids under gravity loading

For partially filled structures the liquid can be treated as incompressible, see [5], [6], [9]. The pressure distribution is given by the hydrostatic pressure law, with ρ as the constant density, $\mathbf{g}$ as the gravity and with the difference of the upper liquid level ${}^{o}\mathbf{x}$ and an arbitrary point $\mathbf{x}$ on the wetted structure. A conservative description is achieved, if the volume conservation of the liquid is taken into account during the deformation of liquid and structure, too.

$$ \begin{aligned} {}^{g}p &= \rho\mathbf{g}\cdot({}^{o}\mathbf{x}-\mathbf{x}) && (5)\\ \text{and}\quad v &= const. && (6) \end{aligned} $$

2.4 Compressible hydrostatic loading- Compressible fluids under gravity loading

A further important case is the composition of dead weight and compressibility of fluid ($i = kg$), see [7]. The corresponding pressure law for technical applications can be found by combining Hooke's law and mass conservation with the assumption of an uniform density distribution throughout the liquid. The hydrostatic pressure law for compressible liquids can be derived from a variational analysis of the gravity potential and the virtual work expression of the pressure resulting from Hooke's law.

$$ \begin{aligned} {}^{kg}p &= {}^{c}p - {}^{x}p - {}^{h}p && (7)\\ &= \rho(v)\mathbf{g}\cdot(\mathbf{c}-\mathbf{x}) - {}^{h}p && (8)\\ \text{with}\quad \rho(v)v &= const. && (9) \end{aligned} $$

${}^{c}p = \rho(v)\mathbf{g}\cdot\mathbf{c}$ is the pressure at the center $\mathbf{c}$ of volume, ${}^{x}p = \rho(v)\mathbf{g}\cdot\mathbf{x}$ denotes the pressure at an arbitrary point $\mathbf{x}$ on the wetted structure. In the view of a mesh-free representation of the fluid, the constitutive equations are described by the shape and the volume of the structure.

2.5 Boundary integral representation of volume and center of volume

The goal of this approach is that all necessary quantities can be expressed by a boundary integral representation. This allows to formulate all state variables via an integration of the surrounding wetted surface. The fluid volume v and the center $\mathbf{c}$ of the volume can be computed via:

$$v = \frac{1}{3}\int_\eta \int_\xi \mathbf{x} \cdot^* \mathbf{n}\; d\xi d\eta \quad (10)$$

$$\text{and} \quad \mathbf{c} = \frac{1}{4v}\int_\eta \int_\xi \mathbf{x}\,\mathbf{x} \cdot^* \mathbf{n}\; d\xi d\eta \quad (11)$$

An large deformation analysis of the structure including the fluid can be performed using a Newton type scheme by applying a Taylor series expansion on the governing equations. The following linearization is restricted to compressible fluids under gravity. All further fluid modells are described in detail in [1], [5], [8].

3 LINEARIZATION FOR COMPRESSIBLE FLUIDS UNDER GRAVITY

Within the Newton scheme the deformed state is computed iteratively. Both, the virtual expression and the additional constraint equations have to be consistently linearized. The linearization of the virtual work expression leads to three parts, the residual part $\delta^{kg}\Pi_t$, the pressure level part $\delta^{kg}\Pi^{\Delta p}$ and the follower force part $\delta^{kg}\Pi^{\Delta n}$.

$$\begin{aligned}\delta^{kg}\Pi_{lin} &= \delta^{kg}\Pi_t + \delta^{kg}\Pi^{\Delta p} + \delta^{kg}\Pi^{\Delta n} && (12)\\ &= \int_\eta \int_\xi ({}^{kg}p_t\,{}^*\mathbf{n}_t + \Delta^{kg}p^*\mathbf{n}_t + {}^{kg}p_t\Delta^*\mathbf{n}) \cdot \delta\mathbf{u}\; d\xi d\eta && (13)\end{aligned}$$

The follower force part is given by the structural displacement $\Delta\mathbf{u}$ along the covariant coordinates of the structure. The pressure change is decomposed into a change of the pressure in the center $\Delta^c p$, the pressure at an arbitrary point $\Delta^x p$ and the compression level $\Delta^h p$.

$$\begin{aligned}\Delta^*\mathbf{n} &= \Delta\mathbf{u},_\xi \times \mathbf{x}_t,_\eta + \mathbf{x}_t,_\xi \times \Delta\mathbf{u},_\eta && (14)\\ \Delta^{kg}p &= \Delta^c p - \Delta^x p - \Delta^h p && (15)\end{aligned}$$

After an integration by parts the linearized equations can be separated into a field and a boundary value problem. The boundary value part vanishes completely for closed structures. The different pressure changes lead to:

$$\begin{aligned}\Delta^c p &= -2\frac{{}^c p_t}{v_t}\int_\eta \int_\xi {}^*\mathbf{n}_t \cdot \Delta\mathbf{u}\; d\xi d\eta + \int_\eta \int_\xi \frac{{}^x p_t}{v_t}\,{}^*\mathbf{n}_t \cdot \Delta\mathbf{u}\; d\xi d\eta && (16)\\ \Delta^x p &= -\frac{{}^x p_t}{v_t}\int_\eta \int_\xi {}^*\mathbf{n}_t \cdot \Delta\mathbf{u}\; d\xi d\eta + \rho_t \mathbf{g} \cdot \Delta\mathbf{u} && (17)\\ \Delta^h p &= -\frac{K}{V}\int_\eta \int_\xi {}^*\mathbf{n}_t \cdot \Delta\mathbf{u}\; d\xi d\eta && (18)\end{aligned}$$

The summary of the pressure changes introduced into the linearized virtual work expression results in a symmetric displacement formulation. This implies that the proposed model is conservative.

$$
\begin{aligned}
\delta^{kg}\Pi_{lin} = \quad & \delta^{kg}\Pi_t \\
+ \quad & \left(\frac{K}{V} - 2\frac{{}^{c}p_t}{v_t}\right)\int_\eta\int_\xi \Delta\mathbf{u}\cdot\ {}^{*}\mathbf{n}_t\ d\xi d\eta \int_\eta\int_\xi {}^{*}\mathbf{n}_t\cdot\delta\mathbf{u}\ d\xi d\eta \qquad \text{part I} \\
+ \quad & \int_\eta\int_\xi \frac{{}^{x}p_t}{v_t}\ {}^{*}\mathbf{n}_t\cdot\Delta\mathbf{u}\ d\xi d\eta \int_\eta\int_\xi {}^{*}\mathbf{n}_t\cdot\delta\mathbf{u}\ d\xi d\eta \\
+ \quad & \int_\eta\int_\xi {}^{*}\mathbf{n}_t\cdot\Delta\mathbf{u}\ d\xi d\eta \int_\eta\int_\xi \frac{{}^{x}p_t}{v_t}\ {}^{*}\mathbf{n}_t\cdot\delta\mathbf{u}\ d\xi d\eta \qquad \text{part II} \\
- \quad & \frac{\rho_t}{2}\int_\eta\int_\xi \delta\mathbf{u}\cdot({}^{*}\mathbf{n}_t\otimes\mathbf{g}+\mathbf{g}\otimes\ {}^{*}\mathbf{n}_t)\Delta\mathbf{u}\ d\xi d\eta \qquad \text{part III} \\
+ \quad & \frac{1}{2}\int_\eta\int_\xi {}^{kg}p_t \begin{pmatrix}\delta\mathbf{u}\\ \delta\mathbf{u}_{,\xi}\\ \delta\mathbf{u}_{,\eta}\end{pmatrix}\cdot\begin{pmatrix}0 & \underline{\mathbf{W}}^{\xi} & \underline{\mathbf{W}}^{\eta}\\ \underline{\mathbf{W}}^{\xi T} & 0 & \\ \underline{\mathbf{W}}^{\eta T} & 0 & 0\end{pmatrix}\begin{pmatrix}\Delta\mathbf{u}\\ \Delta\mathbf{u}_{,\xi}\\ \Delta\mathbf{u}_{,\eta}\end{pmatrix} d\xi d\eta \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{part IV}
\end{aligned} \tag{19}
$$

The different linearized parts can be interpreted as follows:

- I The multiplication of the two surface integrals indicates the volume dependence of the compression level and of the pressure at the center of the fluid volume.
- II The change of the local pressure is influenced by changes of the total volume and changes of the location of the center of the volume.
- III A hydrostatic pressure generates a nonuniform pressure field, which is represented by a symmetric field equation under realistic boundary conditions, see [10], [11], [2].
- IV Follower forces create a symmetric field equation too, considering realistic boundary conditions, see [10], [11], [2], [4], [12].

4 FE-DISCRETIZATION AND SOLUTION ALGORITHM

The virtual work expression

$$
\delta W \quad = \quad \delta^{el}V + \delta^{kg}\Pi - \delta^{ex}\Pi = 0 \tag{20}
$$

followed by a linearization process leads to a residual and a linear term. This have to be discretized with standard FE shell, membrane or continuum elements. Further, the discretized constraint equations as Hooke's law, the mass conservation of the fluid and the computation of the pressure at the center of the structure have to be included, resulting in a hybrid system of equations for the coupled problem:

$$
\begin{bmatrix}
{}^{el,kg}\mathbf{K} & -\mathbf{a} & -\mathbf{b} & \mathbf{a}\\
-\mathbf{a}^T & -\frac{K}{V} & 0 & 0\\
-\mathbf{b}^T & 0 & -2\ {}^{c}pv & v\\
\mathbf{a}^T & 0 & v & 0
\end{bmatrix}
\begin{pmatrix}\mathbf{d}\\ \Delta^{k}p\\ \frac{\Delta\rho}{\rho}\\ \Delta^{c}p\end{pmatrix}
=
\begin{pmatrix}{}^{kg}\mathbf{F}\\ 0\\ 0\\ 0\end{pmatrix}. \tag{21}
$$

This symmetric system can be reduced to a conventional symmetric displacement representation with the elastic and load stiffness matrix ${}^{el,kg}\mathbf{K}$, the residual of internal, external and interaction forces ${}^{kg}\mathbf{F}$, the nodal displacement vector $\mathbf{d}$, a volume pressure gradient ${}^{kg}\alpha$ and two rank-one vectors $\mathbf{a}$ and $\mathbf{b}$.

$$[{}^{el,kg}\mathbf{K} + {}^{kg}\alpha\mathbf{a}\otimes\mathbf{a} + \mathbf{b}\otimes\mathbf{a} + \mathbf{a}\otimes\mathbf{b}]\mathbf{d} = {}^{kg}\mathbf{F} \tag{22}$$

This can be interpreted as a symmetric rank-three update of the matrix ${}^{el,kg}\mathbf{K}$ coupling all wetted degrees of freedom together. Applying the Sherman - Morrison Formula an efficient solution can be computed computed by two additional forward-backward substitutions:

$$\mathbf{d}_1 = {}^{el,kg}\mathbf{K}^{-1}\ {}^{kg}\mathbf{F}, \quad \mathbf{d}_2 = {}^{el,kg}\mathbf{K}^{-1}\ \mathbf{a}, \quad \mathbf{d}_3 = {}^{el,kg}\mathbf{K}^{-1}\ \mathbf{b}. \tag{23}$$

The nodal displacement vector for one iteration step is given by a linear combination of the three auxiliary solution vectors: $\mathbf{d} = \mathbf{d}(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$. For further details and the combination with arc-length schemes, we refer to [7].

5 NUMERICAL EXAMPLES

An elastic cylindrical vessel (weightless, elastic modulus $E = 21 \cdot 10^{10}\frac{N}{m^2}$, Poisson's ratio $\nu = 0.3$) with a very thin wall - close to a membrane - is completely filled with water (density $\rho = 1000\frac{kg}{m^3}$, bulk modulus $K = 0.5 \cdot 10^9\frac{N}{m^2}$). In a first load step the vessel is pressurized by 1 bar at the top of the vessel indicating the weight of the plate. In a second step the structure is loaded by a given displacement u_{ext} of the loading plate, see figure (1b).

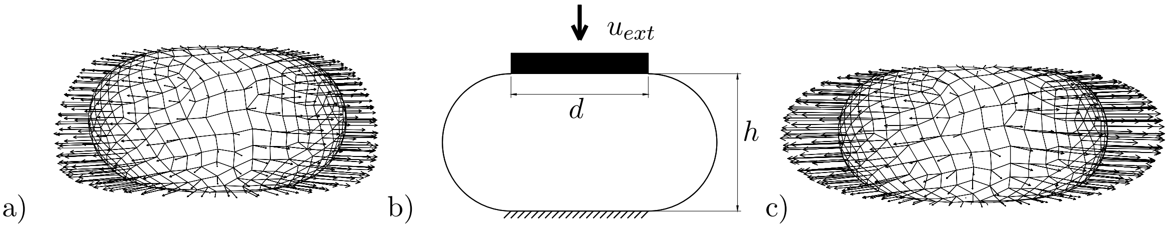

Figure 1: b) Height $h = 10m$, diameter $d = 10m$, piston displacement $u_{ext} = 4m$; a) radial displacement vectors - first load step, c) radial displacement vectors - final load step

Due to the deformation density and volume change according to the conservation of mass, see figure (2) a). The decrease of the volume implicates an increase in the pressure level ${}^{h}p$ in the liquid. For comparison only a gas filling is considered too, see figure (2) b). A provisional result is the position of the center of the volume, which changes with the displacement of the piston, see figure (2) c).

6 CONCLUSION

The proposed approach to describe the gas and fluid effects and their interaction with deforming structures by state equations has several advantages. First, the mesh-free analysis of the fluid resp. the gas avoids remeshing procedures in large deformation analysis. Second, no contact models between fluid and structure have to be considered. Third, stability investigations can be carried out taking the specific decomposition of the stiffness matrix of the coupled problem into account, see [8]. Fourth, the solution of the coupled equation can be efficiently performed based on the subsequent use of the Sherman - Morrison formula involving only the triangular decomposition of the structural matrix. Summarizing all, the computational effort is significant lower and better adjusted than in convential fully discretized methods.

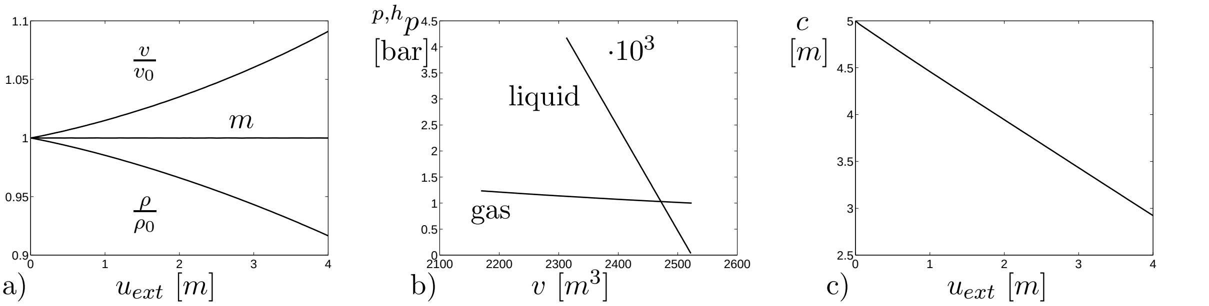

Figure 2: a) mass conservation vs. piston displacement u_{ext}, b) fluid pressure ^{h}p / gas pressure ^{p}p vs. fluid volume v, c) location of center c of volume vs. piston displacement u_{ext}

REFERENCES

[1] J. Bonet, R.D. Wood, J. Mahaney, and P. Heywood. Finite element analysis of air supported membrane structures. *Comput. Methods Appl. Mech. Engrg.*, 190:579–595, 2000.

[2] H. Bufler. Konsistente und nichtkonsistente druckbelastungen durch flüssigkeiten. *ZAMM*, 72(7):T172 – T175, 1992.

[3] H. Bufler. Configuration dependent loading and nonlinear elastomechanics. *ZAMM*, 73:4 – 5, 1993.

[4] G. Romano. Potential operators and conservative systems. *Meccanica*, 7:141–146, 1972.

[5] T. Rumpel and K. Schweizerhof. A hybrid approach for volume dependent fluid-structure-problems in nonlinear static finite element analysis. In J. Eberhardsteiner H.A. Mang, F.G. Rammerstorfer, editor, *Fifth World Congress on Computational Mechanics (WCCM V)*, 2002.

[6] T. Rumpel and K. Schweizerhof. Hydrostatic fluid loading in nonlinear finite element analysis. *Int. J. Numer. Meth. Engng*, 2002/03. submitted /accepted paper.

[7] T. Rumpel and K. Schweizerhof. Large deformation fe analysis of thin walled shell and membrane structures fully filled with compressible fluids. *Int. J. Numer. Meth. Engng*, 2003. in preparation.

[8] T. Rumpel and K. Schweizerhof. Volume-dependent pressure loading and its influence on the stability of structures. *Int. J. Numer. Meth. Engng*, 56:211–238, 2003.

[9] H. Schneider. *Flüssigkeitsbelastete Membrane unter großen Deformationen.* Universität Stuttgart, 1990. Dissertation.

[10] K. Schweizerhof. *Nichtlineare Berechnung von Tragwerken unter verformungsabhängiger Belastung.* Universität Stuttgart, 1982. Dissertation.

[11] K. Schweizerhof and E. Ramm. Displacement dependent pressure loads in nonlinear finite element analyses. *Computers & Structures*, 18(6):1099–1114, 1984.

[12] M. J. Sewell. On configuration-dependent loading. *Arch. Rational Mech. Anal.*, 23:327–351, 1966.