Stability Analysis of Fluid Loaded or Supported Shell Structures

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Abstract

Fluid or gas loadings on thin shell or membrane structures under external loadings (e.g. in oil tanks or gas supported beams) may have a major influence on the structural stability behavior. The goal of this approach, which is based on the works of [2], [3] and [4], is to present some investigation of the influence of such a gas or fluid support (without actually discretizing the fluid/gas) on the eigenvalues and eigenmodes of the stiffness matrix of shell or membrane-like structures undergoing large displacements to allow conclusions concerning stability. For this purpose an efficient algorithm is derived, which benefits from the dyadic rank updates of the stiffness matrix due to volume dependence [1], [2] of the fluid/gas loading.

1 Modal Analysis

Depending on the load case the volume dependence of the fluid or gas pressure leads to several rank updates of the global stiffness matrix $K$ (see [2], [3]). For reasons of simplicity the special case of a single chamber only loaded with an incompressible fluid will serve as an example. The global system matrix $A$ then becomes

$$A = \underbrace{K}_{\text{stiffness part}} + \underbrace{\gamma a a^T}_{\text{volume coupling part}},$$

(1.1)

This dyadic rank update with the volume coupling vector $a$ and a proportional factor $\gamma$ results in a fully occupied part of the system matrix $A$, which may lead to numerical difficulties during the modal decomposition process. To bypass the standard procedure for computing the modal matrix $\Phi$ and the eigenvalues $\chi$ of an almost fully populated system matrix $A$ the modal matrix $\Phi$ is split up into a modal factor matrix $\Xi$ and the modal matrix $\Psi$, which contains all eigenvectors $\psi_i$ of the uncoupled stiffness matrix $K$. Thus $\Phi$ can be written as:

$$\Phi = \Psi \Xi = \left( \begin{array}{c|c|c|c|c} \psi_1 & \psi_2 & \ldots & \psi_i & \ldots & \psi_n \end{array} \right) \left( \begin{array}{c|c|c|c|c} \xi_1 & \xi_2 & \ldots & \xi_i & \ldots & \xi_n \end{array} \right)$$

(1.2)

Introducing the new matrix

$$A^* = \Psi^T A \Psi = \Psi^T \left( K + \gamma a a^T \right) \Psi.$$

(1.3)

and after some rearranging a modified form of the eigenvalue problem $A \Phi = \Phi \chi$ can be given with the columns $\xi_i$ of the modal factor matrix $\Xi$ and the eigenvalues $\chi_i$ as:

$$A^* \xi_i = \chi_i \xi_i.$$

(1.4)
Substituting (1.3) into (1.4) gives us after some reordering
\[
(\Psi^T \left[ K + \gamma_t a a^T \right] \Psi - \chi_i I) \xi_i = 0.
\]
Using the spectral matrix \( \Lambda \) (with its eigenvalues \( \lambda_i \)) of \( K \) along with the modified coupling vector \( \bar{a} = \Psi^T a \) yields
\[
(\Lambda + \gamma_t \bar{a} \bar{a}^T - \chi_i I) \xi_i = 0.
\]

### 1.1 Eigenvalue extraction

For this homogeneous set of equations non-trivial solutions \( \xi_i \neq 0 \) do exist, if the determinant of the coefficient matrix \( (\Lambda + \bar{a} \bar{a}^T - \chi_i I) \) vanishes. As the new eigenvalues are different from the old ones, \( \chi_i \neq \lambda_i \), the unknown eigenvalues \( \chi_i \) can be extracted as solutions of the characteristic polynomial \( p(\chi) \):
\[
p(\chi) = 1 + \gamma_t \sum_{j=1}^{n} \frac{\bar{a}_j \bar{a}_j}{\lambda_j - \chi} = 0,
\]  
with \( \begin{cases} \chi_i > \lambda_i & \text{for } \gamma_t > 0 \\ \chi_i < \lambda_i & \text{for } \gamma_t < 0 \end{cases} \) \( (1.7) \)

Thus the new eigenvalues \( \chi_i \) will increase or decrease depending on the sign of the factor \( \gamma_t \). As already mentioned the polynomial \( p(\chi) \) is strictly monotone between the poles \( \lambda_j \), therefore an efficient method to localize the zeroes in \( p(\chi) \) can be found by the bisection method.

### 1.2 Computation of Eigenvectors

For the computation of the corresponding eigenvectors equation (1.6) is considered again, focusing on the column \( \xi_i \) of the modal factor matrix \( \Xi \). To eliminate the implicit form of \( \xi_i \) it can be normalized by its length, leading to
\[
\xi_i = -\frac{\gamma_t (\Lambda - \chi_i I)^{-1} \bar{a} (\bar{a}^T \cdot \xi_i)}{\|\gamma_t (\Lambda - \chi_i I)^{-1} \bar{a} (\bar{a}^T \cdot \xi_i)\|} = -\frac{(\Lambda - \chi_i I)^{-1} \bar{a}}{\| (\Lambda - \chi_i I)^{-1} \bar{a} \|}
\]
as the \( i^{th} \) column of \( \Xi \). With the multiplicative split (1.2) the transformed modal matrix can then be computed.

### 2 Numerical Example

To show the effect of the fluid support on the stability behavior the development of the eigenvalues of steel cylinder \( (r = 20m, h = 40m, r/t = 1000) \) under axial loading \( \sigma \) will be monitored (see figures 2.1 and 2.2). Increasing the axial loading \( \sigma \) on the empty cylinder (a) ends up at a critical buckling load of \( \sigma_{max} = 7.1 \cdot 10^{-5} N/mm^2 \). The fluid filling of cylinder (b) leads to an increase of the eigenvalues (mainly due to hydrostatic pretensioning) and the subsequent axial loading leads to a critical buckling load of \( \sigma_{max} = 8.3 \cdot 10^{-5} N/mm^2 \), which is about 16% higher than for the empty cylinder. Further on the volume coupling had only small effects on the eigenvalues of the cylinder because the deformations in this example are relatively small. Other examples of soft structures undergoing large deformations showed a clear stiffening due to volume dependence. A stronger effect of the volume coupling on stiffer structures, as e.g. the steel cylinder, is expected for a cylinder completely filled with a compressible heavy fluid, which will be part of further investigations.
References


