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A COMPUTATIONAL FRAMEWORK FOR LARGE DEFORMATION PROBLEMS IN FLEXIBLE MULTIBODY DYNAMICS

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Abstract. *Computer methods for flexible multibody dynamics that are able to treat large deformation phenomena are important for specific applications such as contact problems. From a mechanical point of view, large deformation phenomena are formulated in the framework of nonlinear continuum mechanics. Computer methods for large deformation problems typically rely on the nonlinear finite element method.*

On the other hand classical formalisms for multibody dynamics are based on rigid bodies. Their extension to flexible multibody systems is typically restricted to linear elastic behavior.

In the present work the nonlinear finite element method is extended such that the simulation of flexible multibody dynamics including large deformation phenomena can be handled successfully.

1 INTRODUCTION

In the present work we address computer methods that can handle large deformations in the context of multibody systems. In particular, the link between nonlinear continuum mechanics and multibody systems is facilitated by a specific formulation of rigid body dynamics [1]. This formulation is closely related to the notion of natural coordinates [2]. Our approach makes possible the incorporation of state-of-the-art computer methods for large deformation problems. Examples are arbitrary constitutive models [3], geometrically exact beams and shells [4], domain decomposition [5], and large deformation contact [6].

Energy and momentum consistent numerical methods for this kind of problems offer superior stability and robustness properties. Our approach relies on a uniform formulation of discrete mechanical systems such as rigid bodies and semi-discrete flexible bodies resulting from a finite element discretization of the underlying nonlinear continuum formulation. The uniform formulation results in discrete equations of motion assuming the form of differential-algebraic equations (DAEs). A constant inertia matrix is a characteristic feature of the present DAEs. In particular, the simple DAE structure makes possible the design of structure-preserving time-stepping schemes such as energy-momentum schemes and momentum-symplectic integrators [7, 8].

A further advantage of the present treatment of flexible and rigid bodies is that flexible multibody systems can be implemented in a very systematic way. In fact, the present approach leads to a generalization of the standard finite element assembly procedure. The generalized assembly procedure makes possible the incorporation of both arbitrary nonlinear finite element formulations and multibody features such as joints.

On the other hand the nonstandard description of rigid bodies requires some care concerning the consistent application of actuating forces. This issue will be addressed in the context of kinematic pairs. Moreover the incorporation of large deformation phenomena into the present description of multibody dynamics will be dealt with. In particular, we will outline the inclusion of large deformation contact into flexible multibody dynamics.

2 ROTATIONLESS FORMULATION

The rotationless rigid body formulation [1] relies on redundant coordinates that can be viewed as natural coordinates. Natural coordinates are comprised of Cartesian components of unit vectors and Cartesian coordinates (see [2] and the references cited therein). Let $\mathcal{B} \in \mathbb{R}^3$ be an arbitrary region to be regarded as rigid body. A material point is denoted by $\boldsymbol{X} \in \mathcal{B}$ and the mass density is given by $\rho(\boldsymbol{X})$. The motion of the rigid body can be characterized by a rigid transformation with:

$$\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{\varphi}(t) + \mathbf{R}(t)\,\boldsymbol{X} \qquad . \tag{1}$$

Here $\boldsymbol{\varphi} \in \mathbb{R}^3$ specifies the placement of a point of reference whereas $\mathbf{R} \in \mathsf{SO}(3)$ is a rotation matrix. The column vectors $\boldsymbol{d}_i$ of the rotation matrix $\mathbf{R}$ are denoted as "directors"

$$\boldsymbol{d}_i = \mathbf{R}\,\boldsymbol{e}_i \tag{2}$$

which represent a body-fixed orthonormal frame. The orthonormality condition is enforced with explicit constraints $\Phi_{ij}^{\text{int}} = \boldsymbol{d}_i \cdot \boldsymbol{d}_j - \delta_{ij}$ and associated Lagrange multipliers Λ_{ij}. Accordingly, the vector of redundant coordinates reads $\boldsymbol{q} = (\boldsymbol{\varphi}, \boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3)$. The equations of motion pertaining to the constrained mechanical system at hand can be written as:

$$\begin{aligned} \dot{\boldsymbol{q}} - \boldsymbol{v} &= \boldsymbol{0} \\ \mathbf{M}\,\dot{\boldsymbol{v}} + \nabla_q V - \boldsymbol{f} + \nabla_q \boldsymbol{\Phi}^{\text{int}} : \boldsymbol{\Lambda}^{\text{int}} &= \boldsymbol{0} \\ \Phi_{ij}^{\text{int}} &= 0 \end{aligned} \tag{3}$$

with constant mass matrix $\mathbf{M}$, potential function $V(\boldsymbol{q})$ and generalized force vector $\boldsymbol{f}$. Regarding structural elements such as beams and shells which can undergo large deformations and large rigid body rotations, the semi-discrete equations of motion stemming from the underlying rotationless formulation assume a form analogous to the differential-algebraic equations (3). In the case of structural elements the potential function V typically contains contributions of the stored energy in terms of nonlinear strain measures, see [9] and [4] for further details. The underlying DAE structure of the equations of motion (3) makes possible the straightforward extension to multibody systems.

3 JOINT FORMULATION

Based on previous works [10, 13] we further develop the formulation of joint (or external) constraints. In particular, we develop a modified methodology for the formulation of joint constraints which will be beneficial to the application of external torques. The present approach is unconditionally free of singularities and automatically satisfies the constraints in the initial configuration $\mathcal{B}_0$. Additionally, it provides advantages for the consistent computer implementation.

Let there be two rigid bodies I and II with associated coordinates ${}^{\text{I}}\boldsymbol{\varphi}, {}^{\text{I}}\boldsymbol{d}_i$ and ${}^{\text{II}}\boldsymbol{\varphi}, {}^{\text{II}}\boldsymbol{d}_i$, $i \in \{1,2,3\}$. The two bodies are assumed to be interconnected by a revolute joint. The joint rotation axis will be determined in the initial configuration $\mathcal{B}_0$ by a position vector ${}^{\text{Z}}\boldsymbol{\varphi}$ and a unit vector ${}^{\text{Z}}\boldsymbol{d}_1$. For completeness, we introduce two arbitrary directors ${}^{\text{Z}}\boldsymbol{d}_2$ and ${}^{\text{Z}}\boldsymbol{d}_3$ which fulfill the condition ${}^{\text{Z}}\mathbf{R} = \sum {}^{\text{Z}}\boldsymbol{d}_i \otimes \boldsymbol{e}_i \in \mathsf{SO}(3)$. For each body α with $\alpha \in \{\text{I}, \text{II}\}$ we introduce a body-fixed frame $\{{}^{\alpha}\boldsymbol{\varphi}', {}^{\alpha}\boldsymbol{d}_i'\}$ with:

$$\begin{aligned} {}^{\alpha}\boldsymbol{\varphi}' &= {}^{\alpha}\boldsymbol{\varphi} + {}^{\alpha}\mathbf{R}\,\boldsymbol{c}_{\alpha} \\ {}^{\alpha}\mathbf{R}' &= {}^{\alpha}\mathbf{R}\,\mathbf{C}_{\alpha} \qquad , \end{aligned} \tag{4}$$

$\mathbf{R} = \sum \boldsymbol{d}_i \otimes \boldsymbol{e}_i$ and the constant vectors/matrices $\boldsymbol{c}_{\alpha}$ and $\mathbf{C}_{\alpha}$, which are to be determined in the initial configuration $\mathcal{B}_0$:

$$\begin{aligned} \boldsymbol{c}_{\alpha} &= {}^{\alpha}\mathbf{R}_0^{-1}\,({}^{\alpha}\boldsymbol{\varphi}_0' - {}^{\alpha}\boldsymbol{\varphi}_0) \\ \mathbf{C}_{\alpha} &= {}^{\alpha}\mathbf{R}_0^{-1}\,{}^{\alpha}\mathbf{R}_0' \qquad . \end{aligned} \tag{5}$$

To parameterize the revolute pair, we choose for both bodies the following conditions in the initial configuration $\mathcal{B}_0$:

$$\begin{aligned} {}^{\alpha}\boldsymbol{\varphi}_0' &= {}^{\text{Z}}\boldsymbol{\varphi} \\ {}^{\alpha}\mathbf{R}_0' &= \sum {}^{\text{Z}}\boldsymbol{d}_i \otimes \boldsymbol{e}_i \qquad . \end{aligned} \tag{6}$$

By using the transformed frames $\{{}^{\alpha}\boldsymbol{\varphi}', {}^{\alpha}\boldsymbol{d}_i'\}$, the external constraints corresponding to the revolute joint can now be written as:

$$\boldsymbol{\Phi}^{\text{rev}} = \begin{bmatrix} {}^{\text{I}}\boldsymbol{\varphi}' - {}^{\text{II}}\boldsymbol{\varphi}' \\ {}^{\text{I}}\boldsymbol{d}_1' \cdot {}^{\text{II}}\boldsymbol{d}_2' \\ {}^{\text{I}}\boldsymbol{d}_1' \cdot {}^{\text{II}}\boldsymbol{d}_3' \end{bmatrix} \qquad . \tag{7}$$

It can be concluded that using the transformed frames allows a simple and systematic formulation of any lower kinematic pair with all the advantages mentioned above.

4 CONSISTENT FORMULATION OF TORQUES

Next consider a rigid body on which an external torque $\boldsymbol{M}^*(t) \in \mathbb{R}^3$ is applied. The corresponding expression for the virtual work G^{ext} is given by

$$G^{\text{ext}} = \delta\boldsymbol{\theta}(t) \cdot \boldsymbol{M}^*(t) \qquad . \tag{8}$$

where $\delta\boldsymbol{\theta} \in \mathbb{R}^3$ is a virtual rotation. In the rotationless formulation external torques can be formulated as follower loads. Correspondingly

$$G^{\text{ext}} = \frac{1}{2}\,\delta\boldsymbol{d}_i(t) \cdot (\boldsymbol{M}^*(t) \times \boldsymbol{d}_i(t)) \qquad . \tag{9}$$

We will show that equation (9) is equivalent to equation (8): From the unit-length condition on the directors, $\boldsymbol{d}_i \cdot \boldsymbol{d}_i = 1$, it follows that $\delta\,(\boldsymbol{d}_i \cdot \boldsymbol{d}_i) = 2\,\delta\boldsymbol{d}_i \cdot \boldsymbol{d}_i = 0$. Accordingly, $\delta\boldsymbol{d}_i \perp \boldsymbol{d}_i$ and therefore $\delta\boldsymbol{d}_i = \delta\boldsymbol{\theta} \times \boldsymbol{d}_i$. Then equation (9) directly leads to

$$G^{\text{ext}} = \tfrac{1}{2}\,(\delta\boldsymbol{\theta} \times \boldsymbol{d}_i) \cdot (\boldsymbol{M}^* \times \boldsymbol{d}_i) \qquad . \tag{10}$$

The last equation can be recast in the form

$$\begin{aligned} G^{\text{ext}} &= \tfrac{1}{2}\,\left[(\delta\boldsymbol{\theta} \cdot \boldsymbol{M}^*)\,(\boldsymbol{d}_i \cdot \boldsymbol{d}_i) - (\boldsymbol{d}_i \cdot \boldsymbol{M}^*)\,(\boldsymbol{d}_i \cdot \delta\boldsymbol{\theta})\right] \\ &= \delta\boldsymbol{\theta} \cdot \boldsymbol{M}^* \end{aligned} \tag{11}$$

where the mutual orthonormality of the directors has been taken into account. This corroborates that equation (9) is the appropriate formulation for the inclusion of external torques in the rotationless formulation.

Consistency in the discrete setting Next we show that the naive use of equation (9) in the time discretization in general violates the balance law for angular momentum. Consider a single external torque $\boldsymbol{M}^*$ exerted on the rigid body. In what follows we restrict our attention to the rotational motion of the rigid body. The corresponding equations of motion can be written as

$$G = \delta\boldsymbol{d}_i \cdot \left[M_{ij}\,\ddot{\boldsymbol{d}}_j - \tfrac{1}{2}\boldsymbol{M}^* \times \boldsymbol{d}_i + \nabla_d\boldsymbol{\Phi}^{\text{int}} : \boldsymbol{\Lambda}^{\text{int}}\right] \tag{12}$$

where M_{ij} denote the constant components of the mass matrix. We now make use of a mid-point-type discretization:

$$G_h = \delta\boldsymbol{d}_i \cdot \left[\frac{1}{\Delta t}M_{ij}\left(\dot{\boldsymbol{d}}_{j\,n+1} - \dot{\boldsymbol{d}}_{j\,n}\right) - \tfrac{1}{2}\boldsymbol{M}^*_{n+\frac{1}{2}} \times \boldsymbol{d}_{i\,n+\frac{1}{2}} + \nabla_d\boldsymbol{\Phi}^{\text{int}}\big|_{n+\frac{1}{2}} : \boldsymbol{\Lambda}^{\text{int}}_{n+1}\right] \tag{13}$$

Introducing admissible variations of the form $\delta\boldsymbol{d}_i = \boldsymbol{u} \times \boldsymbol{d}_{i\,n+\frac{1}{2}}$ for arbitrary $\boldsymbol{u} \in \mathbb{R}^3$, equation (13) yields

$$G_h = \boldsymbol{u} \cdot \left[\frac{1}{\Delta t}M_{ij}\left(\boldsymbol{d}_{i\,n+\frac{1}{2}} \times \dot{\boldsymbol{d}}_{j\,n+1} - \boldsymbol{d}_{i\,n+\frac{1}{2}} \times \dot{\boldsymbol{d}}_{j\,n}\right) - \tfrac{1}{2}\boldsymbol{d}_{i\,n+\frac{1}{2}} \times \left(\boldsymbol{M}^*_{n+\frac{1}{2}} \times \boldsymbol{d}_{i\,n+\frac{1}{2}}\right)\right] \;. \tag{14}$$

The definition $M_{ij}\left(\boldsymbol{d}_{i\,n+\frac{1}{2}} \times \dot{\boldsymbol{d}}_{j\,n+1} - \boldsymbol{d}_{i\,n+\frac{1}{2}} \times \dot{\boldsymbol{d}}_{j\,n}\right) = \boldsymbol{L}_{n+1} - \boldsymbol{L}_n$ for the incremental change of the time-discrete angular momentum (see [1]) leads to the following discrete balance equation:

$$\boldsymbol{L}_{n+1} - \boldsymbol{L}_n = -\tfrac{\Delta t}{2}\boldsymbol{d}_{i\,n+\frac{1}{2}} \times \left(\boldsymbol{d}_{i\,n+\frac{1}{2}} \times \boldsymbol{M}^*_{n+\frac{1}{2}}\right) \qquad . \tag{15}$$

The last equation can be alternatively written as

$$\boldsymbol{L}_{n+1} - \boldsymbol{L}_n = -\tfrac{\Delta t}{2}\hat{\boldsymbol{d}}_{i\,n+\frac{1}{2}}\,\hat{\boldsymbol{d}}_{i\,n+\frac{1}{2}}\,\boldsymbol{M}^*_{n+\frac{1}{2}} \qquad . \tag{16}$$

where $\hat{\boldsymbol{d}}$ denotes a skew-symmetric matrix with associated axial vector $\boldsymbol{d} \in \mathbb{R}^3$, so that $\hat{\boldsymbol{d}}\,\boldsymbol{a} = \boldsymbol{d} \times \boldsymbol{a}$ for any $\boldsymbol{a} \in \mathbb{R}^3$. Since in the present time discretization the algebraic constraints are only satisfied at the time nodes, the fulfillment of the orthonormality condition on the directors is restricted to the time nodes as well:

$$\boldsymbol{d}_{i\,n+\alpha} \cdot \boldsymbol{d}_{j\,n+\alpha} \begin{cases} = \delta_{ij} \text{ for } \alpha \in \{0,1\} \\ \neq \delta_{ij} \text{ for } \alpha \in (0,1) \end{cases} \quad \rightarrow \quad \hat{\boldsymbol{d}}_{i\,n+\alpha}\,\hat{\boldsymbol{d}}_{i\,n+\alpha} \begin{cases} = -2\,\mathbf{I} \text{ for } \alpha \in \{0,1\} \\ \neq -2\,\mathbf{I} \text{ for } \alpha \in (0,1) \end{cases} \tag{17}$$

where

$$\boldsymbol{d}_{i\,n+\alpha} = (1-\alpha)\boldsymbol{d}_{i\,n} + \alpha \boldsymbol{d}_{i\,n+1} \tag{18}$$

Therefore, discrete balance of angular momentum $\boldsymbol{L}_{n+1} - \boldsymbol{L}_n = \Delta t \boldsymbol{M}^*_{n+\frac{1}{2}}$ is generally not fulfilled in equation (16). To eliminate this inconsistency, we introduce contravariant directors $\boldsymbol{d}^i_{n+\alpha}$ defined by

$$\boldsymbol{d}^i_{n+\alpha} \cdot \boldsymbol{d}_{j\,n+\alpha} = \delta^i_j \text{ for any } \alpha \qquad . \tag{19}$$

We now define as discrete version of (9):

$$\tilde{G}^{\text{ext}}_h = \frac{1}{2}\,\delta \boldsymbol{d}_i(t) \cdot \left(\boldsymbol{M}^*_{n+\frac{1}{2}} \times \boldsymbol{d}^i_{n+\frac{1}{2}} \right) \qquad . \tag{20}$$

In the following we will refer to this approach as the contravariant torque formulation. It can be easily verified by following the above lines that the contravariant torque formulation restores the fulfillment of the discrete balance law for the angular momentum.

5 COORDINATE AUGMENTATION TECHNIQUE

An alternative approach to the incorporation of torques into the rotationless rigid body formulation relies on the coordinate augmentation technique, see [14]. This approach is based on the introduction of an additional constraint Φ^{aug} in conjunction with an additional rotational degree of freedom γ. Thanks to the systematic approach in section 3, this constraint can be applied directly with:

$$\Phi^{\text{aug}} = {}^{\text{I}}\boldsymbol{d}'_2 \cdot {}^{\text{II}}\boldsymbol{d}'_3 + \sin\gamma + {}^{\text{I}}\boldsymbol{d}'_3 \cdot {}^{\text{II}}\boldsymbol{d}'_3 - \cos\gamma \tag{21}$$

and will be denoted as “classical augmentation”. It belongs to the following class of augmentation constraints

$$\Phi^{\text{aug}} = a\left({}^{\text{I}}\boldsymbol{d}'_2 \cdot {}^{\text{II}}\boldsymbol{d}'_3 + \sin\gamma\right) + b\left({}^{\text{I}}\boldsymbol{d}'_3 \cdot {}^{\text{II}}\boldsymbol{d}'_3 - \cos\gamma\right) \qquad . \tag{22}$$

with constant parameters a and b. It can be shown that for each combination of these parameters there exist configurations $\gamma^* = \arctan\left(\frac{b}{a}\right)$ leading to rank deficiency in the discrete setting. This problem can be alleviated by choosing a and b as fixed quantities at t_n obtained from the previous time step:

$$\begin{aligned} a &= \cos(\gamma_n) \\ b &= \sin(\gamma_n) \qquad . \end{aligned} \tag{23}$$

It can be shown that this modified augmentation is stable up to rotations of $\frac{1}{2}\pi$ per time step. As can be observed from equation (22), the augmentation constraint is nonlinear in the coordinate γ. An energy-momentum consistent time-stepping scheme can be obtained by applying the notion of a discrete gradient in the sense of [15].

6 INCLUSION OF LARGE DEFORMATION CONTACT

The presented framework based on a set of differential-algebraic equations can be directly extended to large-deformation contact problems. In particular, unilateral contact constraints can be formulated as a set of inequality constraints which can be rewritten as equality constraints using a standard active set strategy. The node-to-surface (NTS) method (see [11] for details) can be considered the prevailing method for contact problems in the context of finite elements. Actual developments extend the collocation-type NTS method to a variationally consistent formulation known as mortar contact method (see [12]). For both methods, the classical Karush-Kuhn-Tucker conditions read

$$\Phi^{con} \leq 0, \quad \lambda^{con} \geq 0, \quad \Phi^{con}\lambda^{con} = 0 \tag{24}$$

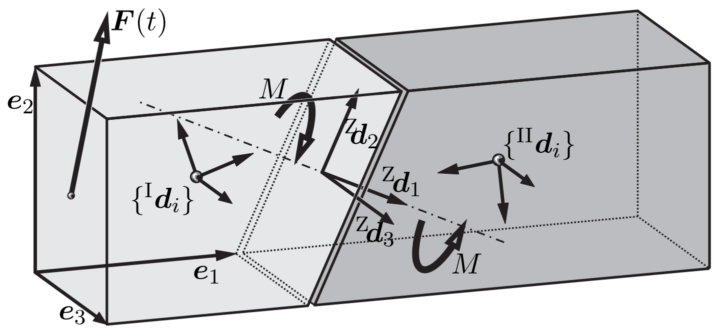
which can be rewritten as

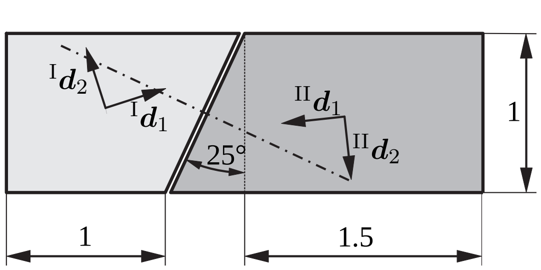
$$\tilde{\Phi}^{con} = \lambda^{con} - \max\{0, \lambda^{con} - c\Phi^{con}\}, \quad c > 0 \tag{25}$$

This formulation makes possible a very efficient computer implementation of the active set strategy. We refer to [6] for a full account on the present formulation of large deformation contact problems.

7 EXAMPLE 1

In the first numerical example we investigate the application of external torques in the rotationless formulation of multibody dynamics. To this end we consider the revolute pair depicted in Fig. 1.





$$\boldsymbol{F}(t) = f(t) \cdot \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$f(t) = \begin{cases} 2\,t & \text{for } 0 < t \leq 0.5 \\ -2\,t + 1 & \text{for } 0.5 < t \leq 1 \\ 0 & \text{for } t > 1 \end{cases}$$

$$M^* = 2$$

Figure 1: Revolute pair: Geometry of the two rigid bodies (prisms) with density $\rho = 1$ and definition of the loads.

The revolute pair consists of two prisms that are connected by a revolute joint with axis ${}^{\mathrm{Z}}\boldsymbol{d}_1$. All dimensions of the kinematic pair, except the thickness in $\boldsymbol{e}_3$-direction which is 1, are given

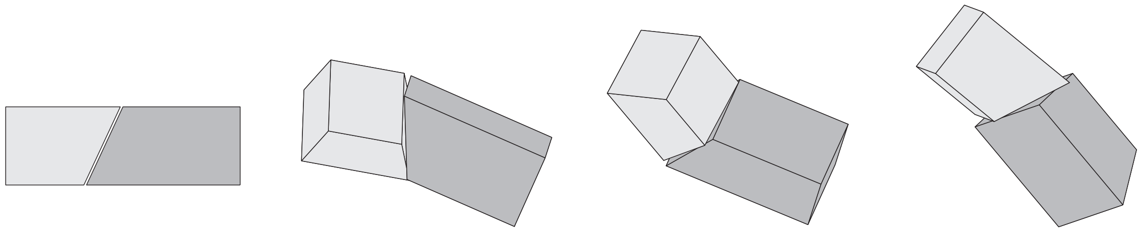


Figure 2: Revolute pair: Snapshots of the motion.

in Fig. 1. To initiate the overall motion of the system, a dead force $\boldsymbol{F}(t)$ is acting for $t < 1$. Throughout the motion a constant internal torque M^* is applied about the revolute axis ${}^{\mathrm{Z}}\boldsymbol{d}_1$. After $t = 1$ no external loads are acting so that total linear and angular momentum have to be conserved quantities.

The resulting motion of the 2-body system is illustrated with some snapshots in Fig. 2. Three alternative ways of applying the actuating torque M^* are considered:

1. Straightforward mid-point evaluation of equation (9), termed naive approach.
2. Newly proposed contravariant torque formulation, see equation (20).
3. Coordinate augmentation outlined in Section 5.

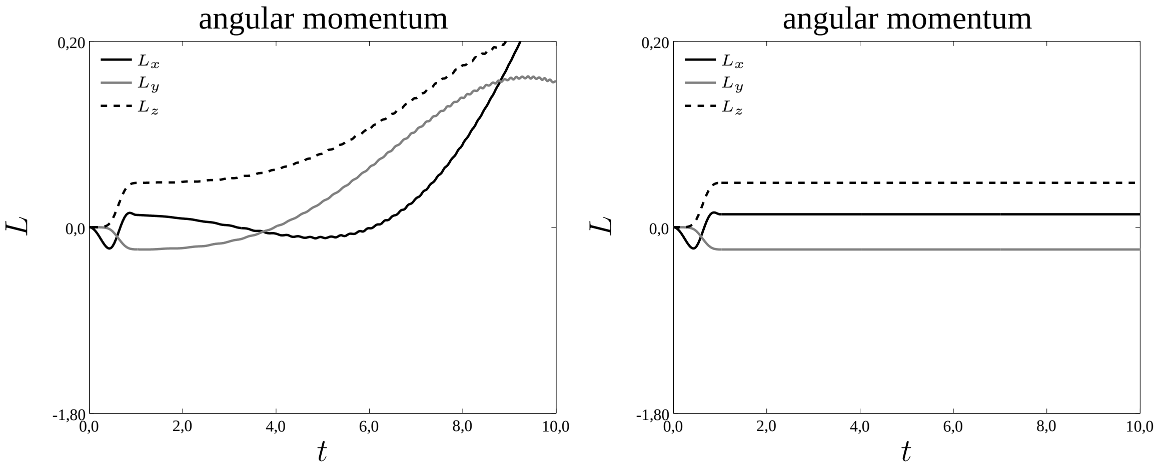
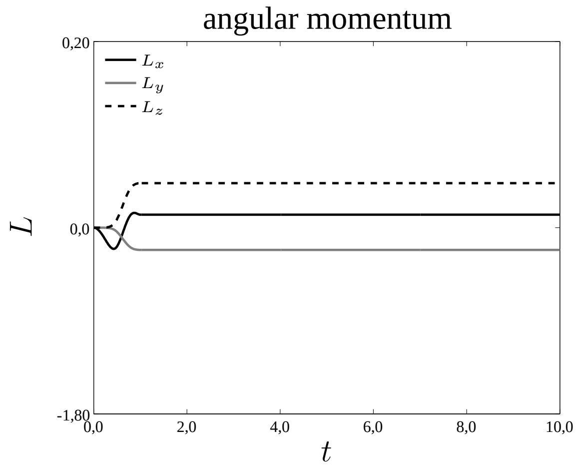


Figure 3: Revolute pair ($\Delta t = 1 \cdot 10^{-2}$): Failure of the naive approach (left) and success of two alternative approaches (right).

It can be observed from Fig. 3 that the naive approach fails to satisfy conservation of angular momentum. In contrast to that, the two alternative methods at hand do fulfill this fundamental conservation law. Although approach 3. guarantees conservation of angular momentum for any time step, it may lead to unphysical behavior depending on the specific formulation of the augmentation constraint. This can be seen from Fig. 4 which depicts the total kinetic energy versus time for two different augmentation constraints. Whereas the "classical augmentation" leads to unphysical growth and decay of the kinetic energy, the alternative method described in Section 5 yields the correct increase in the kinetic energy (see "variable augmentation" in Fig. 4).

It can be concluded that a naive application of torques in the rotationless formulation of multibody dynamics can result in completely unphysical numerical results.



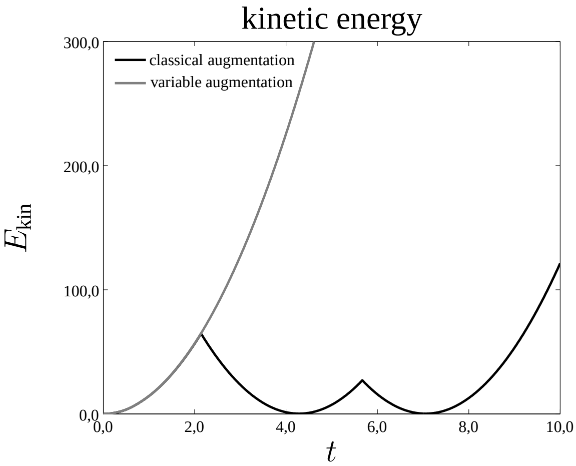


Figure 4: Revolute pair ($\Delta t = 1 \cdot 10^{-2}$): Total angular momentum (left) and total energy (right) for both "classical" and "variable augmentation".

8 EXAMPLE 2

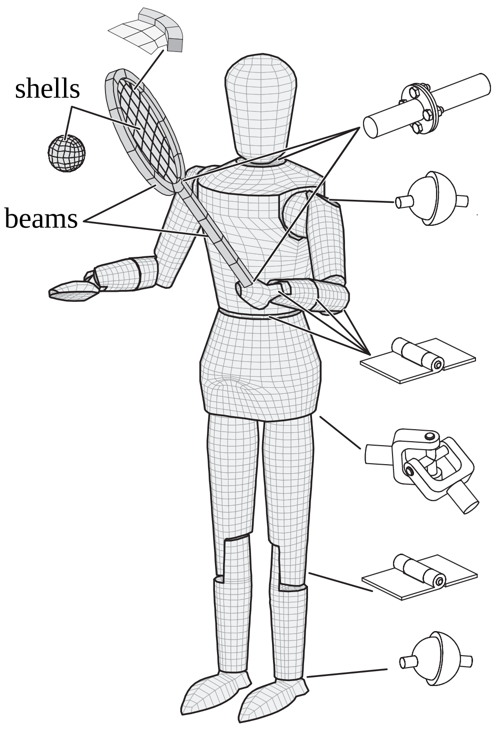


Figure 5: Multibody system with flexible components

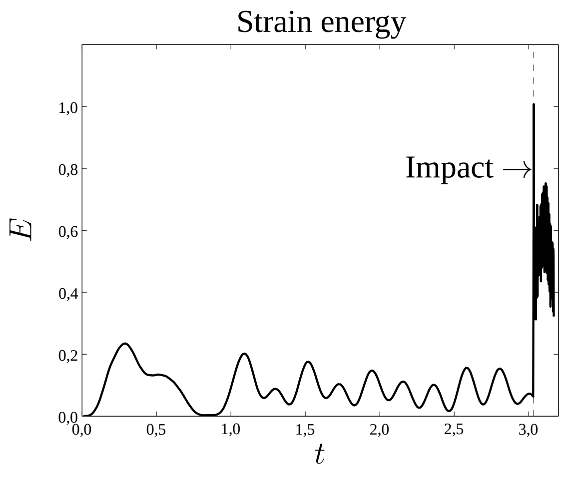


Figure 6: Strain energy of flexible components

The second example deals with the flexible multibody system depicted in Fig. 5. This example demonstrates the inclusion of geometrically exact beams and shells as well as large deformation contact within the framework of flexible multibody dynamics. The model of a tennis player consists of 19 rigid bodies, whereas the tennis racket is modeled with nonlinear beams and shells (see Fig. 5). Shell elements are also used for modeling the tennis ball. The motion of the tennis player himself is prescribed (fully actuated). Due to the presence of the flexible tennis racket the whole system is highly underactuated. The motion of the system until the onset of contact between the tennis ball and the racket is illustrated with some snapshots in Fig. 7. The impact of the tennis ball on the racket leads to large deformations accompanied with a sudden increase of the strain energy (Fig. 6).

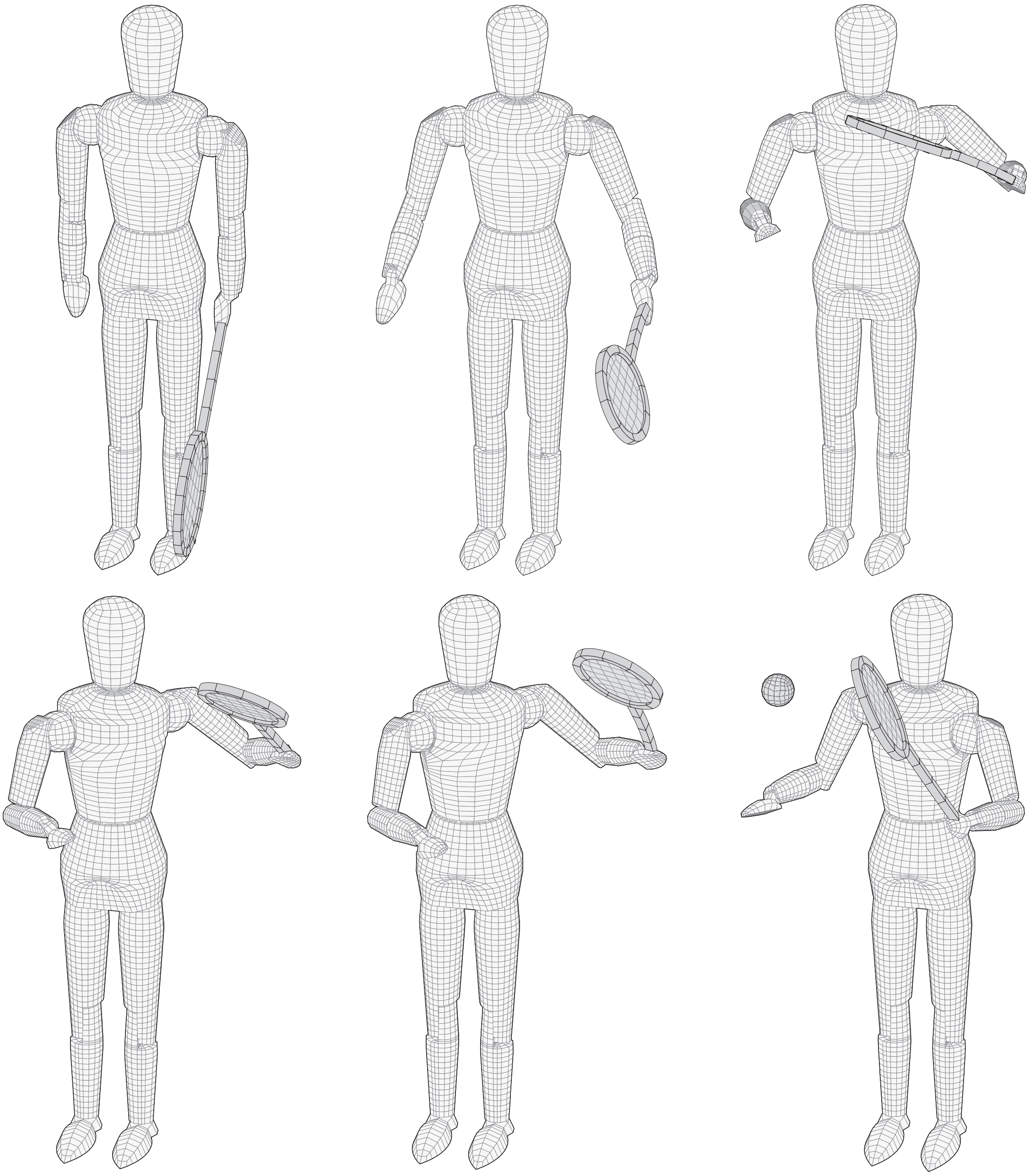


Figure 7: Snapshots of the motion

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