# Application of domain decomposition methods to isogeometric analysis

**Christian Hesch**[1, *] and **Peter Betsch**[1]

[1] Chair of Computational Mechanics, Paul-Bonatz-Str. 9-11, 57078 Siegen

During the past decade various new spatial discretization techniques have been developed. In particular, the usage of NURBS based shape functions, well known to the CAD community, has been adapted to finite element technology. In the present work we use the mortar finite element method for the coupling of nonconforming discretized sub-domains in the framework of nonlinear elasticity. We show that the method can be applied to isogeometric analysis with little effort, once the framework of NURBS based shape functions has been implemented. Furthermore, a specific coordinate augmentation technique allows the design of an energy-momentum scheme for the constrained mechanical system under consideration.



## 1 Fundamental equations

The subsequent development is based on the work by Hughes et al. [1] and Hesch & Betsch [2]. We consider a general non-linear mechanical system, artificially subdivided into several sub-domains $i$, occupying the space $\mathcal{B}_0^{(i)} \subset \mathbb{R}^3$ in the reference configuration. The Lagrangian form of the balance of linear momentum is given by

$$\dot{\boldsymbol{\varphi}}^{(i)} = \rho_0^{-1}\boldsymbol{\pi}^{(i)}, \quad \dot{\boldsymbol{\pi}}^{(i)} = \mathrm{Div}(\boldsymbol{P}^{(i)}) + \boldsymbol{B}^{(i)} \tag{1}$$

supplemented by the boundary conditions

$$\boldsymbol{\varphi}^{(i)} = \bar{\boldsymbol{\varphi}}^{(i)} \quad \text{on} \quad \Gamma_{\boldsymbol{u}}^{(i)} \times [0,T], \quad \boldsymbol{P}^{(i)}\boldsymbol{N}^{(i)} = \bar{\boldsymbol{T}}^{(i)} \quad \text{on} \quad \Gamma_{\boldsymbol{\sigma}}^{(i)} \times [0,T] \tag{2}$$

where $\boldsymbol{P} = 2\boldsymbol{F}\nabla_{\boldsymbol{C}}\Psi(\boldsymbol{C})$ denotes the first Piola-Kirchhoff stress tensor, $\boldsymbol{N}^{(i)}$ the outward unit normal vector in the reference configuration of body $i$, $\boldsymbol{B}^{(i)}$ the applied body forces and $\bar{\boldsymbol{T}}^{(i)}$ the prescribed tractions. Furthermore, $\boldsymbol{\varphi}^{(i)}(\boldsymbol{X}^{(i)}, t) : \mathcal{B}_0 \times [0,T] \to \mathbb{R}^3$ denotes the deformation mapping and $\boldsymbol{C}^{(i)} : \mathcal{B}_0^{(i)} \times [0,T] \to \mathbb{R}^{3\times 3}$ the right Cauchy-Green deformation tensor, given by $\boldsymbol{C}^{(i)} = \boldsymbol{F}^{(i),T}\boldsymbol{F}^{(i)}$, $\boldsymbol{F}^{(i)} = D\boldsymbol{\varphi}^{(i)}$. Setting the $L_2-$inner product on $\mathcal{B}_0^{(i)}$ in the usual fashion

$$\int_{\mathcal{B}_0^{(i)}} (\bullet)\cdot(\bullet)\,\mathrm{d}V^{(i)} =: \langle \bullet, \bullet\rangle^{(i)} \quad \text{and} \quad \int_{\partial\mathcal{B}_0^{(i)}} (\bullet)\cdot(\bullet)\,\mathrm{d}\Gamma^{(i)} =: \langle \bullet, \bullet\rangle_{\Gamma^{(i)}} \tag{3}$$

the contribution of each sub-domain $i$ to the virtual work takes the form

$$\begin{aligned} G^{(i)}(\boldsymbol{\varphi}, \delta\boldsymbol{\varphi}) =& \langle \rho_R \ddot{\boldsymbol{\varphi}}, \delta\boldsymbol{\varphi}\rangle^{(i)} + \langle \boldsymbol{P}, \nabla_{\boldsymbol{X}}\ (\delta\boldsymbol{\varphi})\rangle^{(i)} - \\ & \langle \rho_R \bar{\boldsymbol{B}}, \delta\boldsymbol{\varphi}\rangle^{(i)} - \langle \bar{\boldsymbol{T}}, \delta\boldsymbol{\varphi}\rangle_{\Gamma_{\sigma}^{(i)}} - \langle \boldsymbol{t}, \delta\boldsymbol{\varphi}\rangle_{\Gamma_d^{(i)}} \end{aligned} \tag{4}$$

where $\langle \boldsymbol{t}, \delta\boldsymbol{\varphi}\rangle_{\Gamma_d^{(i)}}$ denotes the virtual work contributions of the coupling tractions. Additionally, we require that the balance of linear momentum across the interface

$$\sum_i -\langle \boldsymbol{t}, \delta\boldsymbol{\varphi}\rangle_{\Gamma_d^{(i)}} = \langle \boldsymbol{t}^{(1)}, (\delta\boldsymbol{\varphi}^{(1)} - \delta\boldsymbol{\varphi}^{(2)})\rangle_{\Gamma_d^{(1)}} = 0 \tag{5}$$

has to hold at all times $t \in [0,T]$.

## 2 Discretization

We apply NURBS based shape functions for the approximation of displacement based finite elements in space

$$\boldsymbol{\varphi}^{(i),\mathrm{h}} = \sum_{A\in\omega^{(i)}} R^{A,(i)}\boldsymbol{q}_A^{(i)}, \quad \delta\boldsymbol{\varphi}^{(i),\mathrm{h}} = \sum_{A\in\omega^{(i)}} R^{A,(i)}\delta\boldsymbol{q}_A^{(i)} \tag{6}$$

* Corresponding author: Email christian.hesch@uni-siegen.de, phone +0049 271 740 2101, fax +0049 271 740 2436

where

$$R^{A,(i)} = R^{\mathrm{i},\mathrm{j},\mathfrak{k}}_{p,q,r}(\boldsymbol{\xi}) = \frac{N_{\mathrm{i},p}(\xi)M_{\mathrm{j},q}(\eta)L_{\mathfrak{k},r}(\zeta)w_{i,j,k}}{\sum_{\hat{\mathrm{i}}=1}^{n}\sum_{\hat{\mathrm{j}}=1}^{m}\sum_{\hat{\mathfrak{k}}=1}^{l} N_{\hat{\mathrm{i}},p}(\xi)M_{\hat{\mathrm{j}},q}(\eta)L_{\hat{\mathfrak{k}},r}(\zeta)w_{\hat{\mathrm{i}},\hat{\mathrm{j}},\hat{\mathfrak{k}}}} \tag{7}$$

Here, $p, q, r$ denotes the order of the non-rational B-Spline shape functions $N$, $M$ and $L$, recursively defined as follows

$$N_{\mathrm{i},p} = \frac{\xi - \xi_{\mathrm{i}}}{\xi_{\mathrm{i}+p} - \xi_{\mathrm{i}}} N_{\mathrm{i},p-\mathrm{i}}(\xi) + \frac{\xi_{\mathrm{i}+p+1} - \xi}{\xi_{\mathrm{i}+p+1} - \xi_{\mathrm{i}+1}} N_{\mathrm{i}+1,p+1}(\xi), \quad N_{\mathrm{i},0}(\xi) = \begin{cases} 1 & \text{if } \xi_{\mathrm{i}} \leq \xi < \xi_{\mathrm{i}+1} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The definition for $M$ and $L$ follows analogously. Furthermore, $w_{\mathrm{i},\mathrm{j},\mathfrak{k}}$ are NURBS weights, for details see Piegl & Tiller [3].

## 3 Mesh-tying constraints

Four corner nodes of each surface element of the interface, independent of the order of the underlying NURBS solid are given, such that we can always apply a linear interpolation of the Lagrange multipliers

$$\boldsymbol{t}^{(1),\mathrm{h}} = \sum_{A \in \tilde{\omega}^{(1)}} N^A \boldsymbol{\lambda}_A \tag{9}$$

Inserting (9) and (6) in (5) yields

$$\boldsymbol{\lambda}_A \cdot \left( n^{AB} \delta \boldsymbol{q}_B^{(1)} - n^{AC} \delta \boldsymbol{q}_C^{(2)} \right) = 0, \quad n^{AB} = \langle N^A, R^B \rangle_{\Gamma_d^{(1)}}, \quad n^{AC} = \langle N^A, R^C \rangle_{\Gamma_d^{(1)}} \tag{10}$$

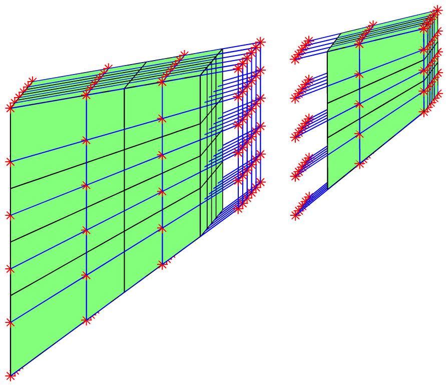
The segment contributions to the constraints reads

$$n^{\kappa\beta} = \langle N^{\kappa}(\boldsymbol{\xi}_{\mathrm{seg}}^{(1),\mathrm{h}}(\boldsymbol{\eta}))R^{\beta}(\boldsymbol{\xi}_{\mathrm{seg}}^{(1),\mathrm{h}}(\boldsymbol{\eta}))\rangle_{\Gamma_{d,\mathrm{seg}}^{(1)}}, \quad n^{\kappa\zeta} = \langle N^{\kappa}(\boldsymbol{\xi}_{\mathrm{seg}}^{(1),\mathrm{h}}(\boldsymbol{\eta}))R^{\zeta}(\boldsymbol{\xi}_{\mathrm{seg}}^{(2),\mathrm{h}}(\boldsymbol{\eta}))\rangle_{\Gamma_{d,\mathrm{seg}}^{(1)}} \tag{11}$$

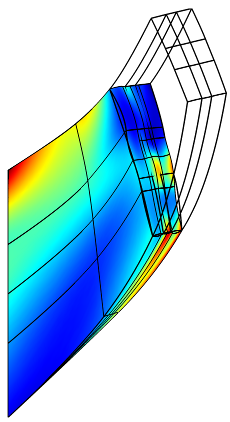
For details, see Hesch & Betsch [2].

## 4 Examples

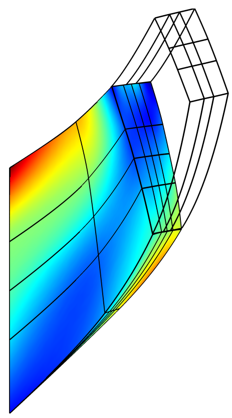
A decomposed and partially h-refined 3D version of Cook's membrane is used to demonstrate the superiority of the mortar method in contrast to the knot-to-surface method (see Temizer et al. [4]). The norm of the Cauchy stresses is displayed below.



Control mesh.



KTS method.



Mortar method.

## References


[1] T.J.R. Hughes, J.A. Cottrell and Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, Comput. Methods Appl. Mech. Engrg., **194**, 4135-4195, (2005).

[2] C. Hesch and P.Betsch, Isogeometric analysis and domain decomposition methods, Comput. Methods Appl. Mech. Engrg. **213-216**, 104-112, (2012).

[3] L. Piegl and W. Tiller. The NURBS Book, Springer, 2nd edition, (2010).

[4] I. Temizer, P. Wriggers, and T.J.R. Hughes, Contact treatment in isogeometric analysis with NURBS, Comput. Methods Appl. Mech. Engrg., **200**, 1100-1112, (2011).