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Structure preserving integrators for non-linear structural dynamics

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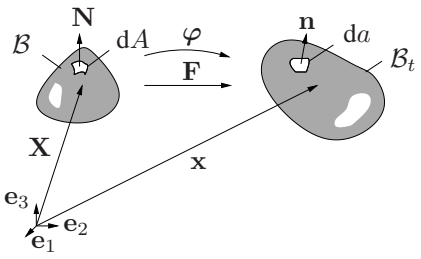
A well known and major drawback of standard time integration schemes in the field of non-linear elastodynamics is their unstable behavior in the case of stiff material behaviour. Even second order accurate implicit time integration schemes are unable to resolve the problem under consideration effectively. To remedy this drawback, structure preserving integrators have been developed. Therefore, the goal of this paper is to compare recently developed integrators. In particular, an energy and momentum conserving scheme, based on a publication by Betsch & Steinmann [1], as well as a symplectic variational integrator, proposed by Lew et al. [4] and Wendlandt & Marsden [3], based on a mid-point evaluation of the discrete Lagrangian, are presented. Two representative numerical examples will outline the characteristics of the different approaches. In particular, a stiff non-linear spring pendulum and a finite element model of non-linear structural dynamics are considered.

1 Continuum mechanics

To describe continuum bodies (see figure), one distinguishes between the reference (left body) and the current configuration $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$ (right body). For the deformation and the strain, the deformation tensor $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ and the right Cauchy-Green strain tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\,\mathbf{F}$ are introduced. As constitutive law, we consider a compressible neo-Hookean model, corresponding to the strain energy function $W(\mathbf{C}) = \frac{\mu}{2}\,(\mathrm{tr}(\mathbf{C} - n_{dim}) + \frac{\lambda}{2}\,\left(\ln(\sqrt{\det(\mathbf{C})})\right)^2 - \mu\,\ln(\sqrt{\det(\mathbf{C})})$.



We further need to calculate the first and second Piola-Kirchhoff stress tensor $\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}$, $\mathbf{S} = 2\,\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}$ as well as the elasticity tensor $\mathbb{D} = 4\,\frac{\partial^2 W(\mathbf{C})}{\partial \mathbf{C} \otimes \partial \mathbf{C}}$. The fundamental balance principles (see tabular on the right) have to be satisfied for all times. In the case of a conservative system without any external forces and torques, the derivative with respect to time of the conserved quantities $(m, \mathbf{J}, \mathbf{L}, H)$ is equal to zero.

Balance principles	
Mass:	$\frac{\mathrm{d}}{\mathrm{d}t} m = 0$
Linear momentum:	$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{J} = \int_{\mathcal{B}} \mathbf{B}\,\rho_0\,\mathrm{d}V + \int_{\partial\mathcal{B}} \mathbf{T}\,\mathrm{d}A$
Angular momentum:	$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{L} = \int_{\mathcal{B}} \boldsymbol{\varphi} \times \mathbf{B}\,\rho_0\,\mathrm{d}V + \int_{\partial\mathcal{B}} \boldsymbol{\varphi} \times \mathbf{T}\,\mathrm{d}A$
Energy:	$\frac{\mathrm{d}}{\mathrm{d}t} H = \dot{T} + \dot{V}$

2 Spatial discretisation

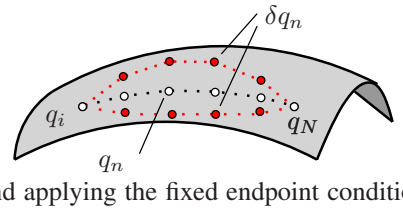
We obtain the weak form of the field equations, by multiplication with a test function $\boldsymbol{\eta}$ and partial integration:

$$\int_{\mathcal{B}} \mathbf{P} : \mathrm{Grad}(\boldsymbol{\eta})\,\mathrm{d}V - \int_{\mathcal{B}} \rho_0\,\left(\mathbf{B} - \dot{\mathbf{V}}\right) \cdot \boldsymbol{\eta}\,\mathrm{d}V - \int_{\partial\mathcal{B}_\sigma} \mathbf{T} \cdot \boldsymbol{\eta}\,\mathrm{d}A = 0$$

The considered continuum bodies will be discretized by the finite element method. Therefore, the bodies are split into a finite number of elements. For this, we use isoparametric finite element interpolations $\mathbf{X}^h(\boldsymbol{\xi}, t) = \sum_{A=1}^{n_{node}} N_A(\boldsymbol{\xi})\,\mathbf{X}^A(t)$. The shape functions are interpolated with Lagrange polynomials $N_A(\xi) = \prod_{\substack{B=1 \\ B \neq A}}^{n} \frac{(\xi_B - \xi)}{(\xi_B - \xi_A)}$.

3 Time discretisation

Variational integrators: The idea is to discretize the action principle. The path $q(t)$ is replaced by a discrete path q_n. The Lagrangian L is replaced by a discrete Lagrangian $L_{\mathrm{d}}(\mathbf{q}_n, \mathbf{q}_{n+1}, h) = h\,L\left((1-\alpha)\,\mathbf{q}_n + \alpha\,\mathbf{q}_{n+1}, \frac{\mathbf{q}_{n+1} - \mathbf{q}_n}{h}\right)$. The action functional $S(q) = \int_a^b L(\mathbf{q}, \dot{\mathbf{q}})\,\mathrm{d}t$ can be approximated with quadrature, leading to the discrete action functional $S_{\mathrm{d}} = \sum_{n=0}^{N-1} L_{\mathrm{d}}(\mathbf{q}_n, \mathbf{q}_{n+1})$.



Variation of the discrete action functional over the whole path, setting it to zero and applying the fixed endpoint conditions $\delta q_i = \delta q_N = 0$ leads to the discrete Euler-Lagrange equations $D_1\,L_d(\mathbf{q}_n, \mathbf{q}_{n-1}) + D_2\,L_d(\mathbf{q}_{n+1}, \mathbf{q}_n) = 0$.

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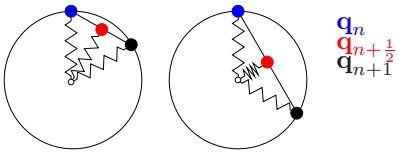
Energy-momentum scheme: Using an energy-momentum scheme, the derivative of the potential is replaced by a so-called discrete gradient in the residual $\frac{2}{h}\,\mathbf{M}\,\left(\mathbf{q}_{n+1}-\mathbf{q}_n\right)-2\,\mathbf{p}_n+h\,\left(\overline{\nabla}_{\mathbf{q}}V_{n+\frac{1}{2}}\right)=0$, as it has been proposed by [1]. In the last equation, the discrete gradient $\overline{\nabla}_{\mathbf{q}}V_{n+\frac{1}{2}}=\bar{\mathbf{F}}^{int}-\bar{\mathbf{F}}^{ext}$ contains the internal forces $\bar{\mathbf{F}}^{int}=\int_{\mathcal{B}}\overline{\nabla}W\left(\mathbf{C}_n,\mathbf{C}_{n+1}\right):\frac{\partial\mathbf{C}(\mathbf{q}_{n+\frac{1}{2}})}{\partial\mathbf{q}_{n+\frac{1}{2}}}\,\mathrm{d}V$. In this connection, we need to calculate the discrete gradient of the strain energy function as follows:

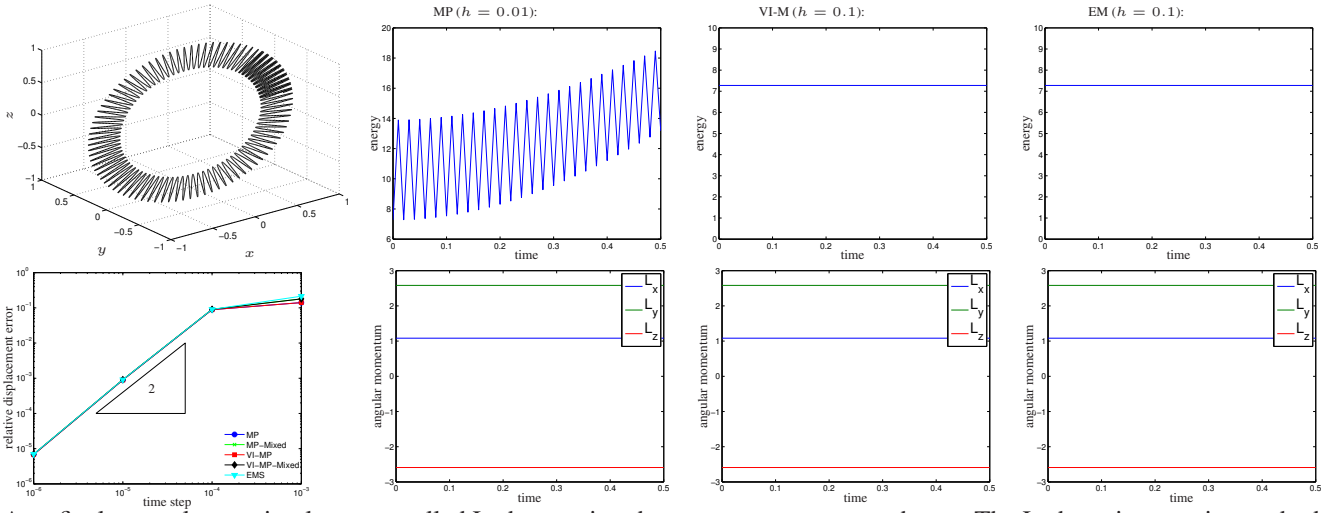
$$\overline{\nabla}W=\nabla W_{n+\frac{1}{2}}+\left(\frac{W_{n+1}-W_n}{\|\mathbf{C}_{n+1}-\mathbf{C}_n\|}-\boldsymbol{\mathcal{M}}:\nabla W_{n+\frac{1}{2}}\right)\boldsymbol{\mathcal{M}},\quad \boldsymbol{\mathcal{M}}=\frac{\mathbf{C}_{n+1}-\mathbf{C}_n}{\|\mathbf{C}_{n+1}-\mathbf{C}_n\|}$$

Now we regard an extension of the mid-point rule applicable for variational integrators as well as for energy-momentum schemes. By applying the mid-point rule and large time steps, a geometric error occurs. The situation is displayed with a spring pendulum on the right side. To remedy this drawback, an average strain $\mathbf{C}_{n+\frac{1}{2}}=\frac{1}{2}\,(\mathbf{C}_{n+1}+\mathbf{C}_n)$ can be used, which can be derived using a three field Hu-Washizu functional.

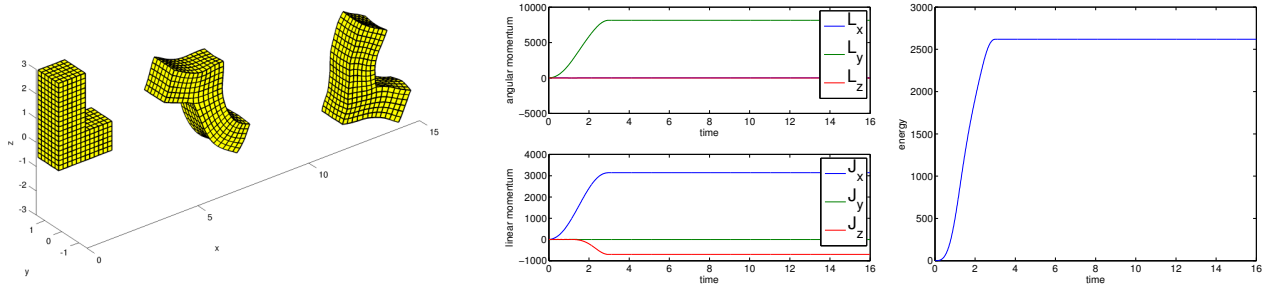


4 Numerical examples

As a first example, we consider a non-linear spring pendulum. The initial conditions are taken from [2]. The relative displacement error after 0.63 seconds has been calculated, based on a solution using a step size of $h=5\cdot10^{-7}$. As a result, we obtain for all integrators a second order accuracy. Varying the initial conditions leads to an energy blow up for the mid-point rule. In comparison to that, the modified variational integrator and the energy-momentum scheme remain stable.



As a final example, we simulate a so-called L-shape using the energy-momentum scheme. The L-shape is a continuum body in form of a three-dimensional L, which is free in space. As material model, a compressible neo-Hookean model is used. A pressure load is applied for three seconds on the left and right surface. The snapshots of the movement are displayed on the left. The angular and linear momentum (middle) and the energy (right) remain constant after removing the pressure load.



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