# Geometrically exact beam theory

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## Introduction

Cosserat beam [1]

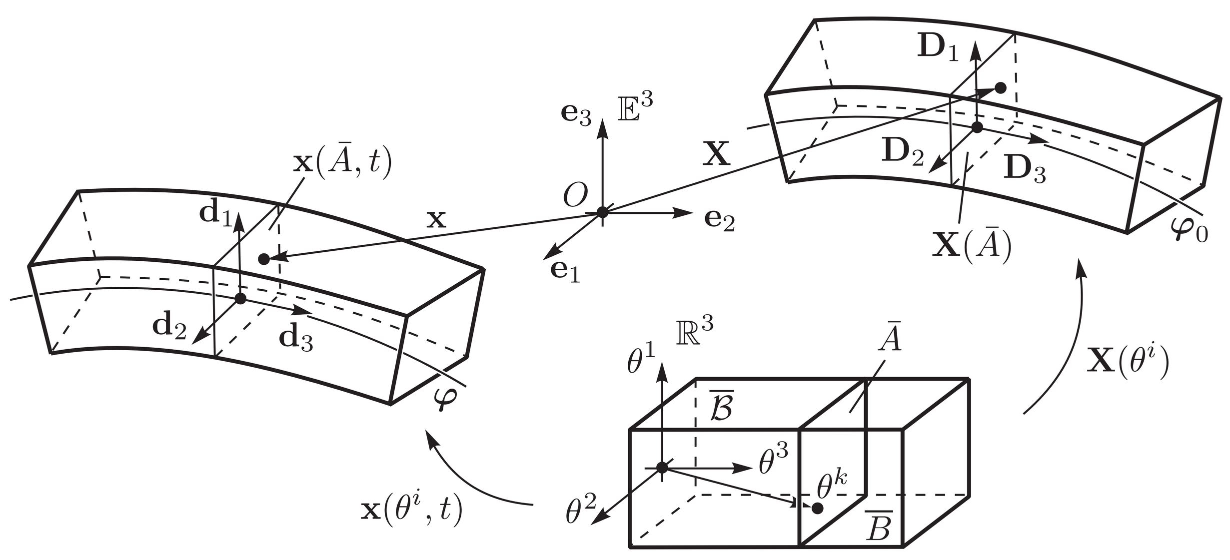
- Nonlinear beam finite elements
- Interpolation of the director field
- Does not rely on rotational degrees of freedom
- Frame indifference and conservation of angular momentum

Director-based theory in skew coordinates [2]

- Formulation in convected coordinates
- Accounts for the lack of orthonormality of the discrete director frame
- Improves dramatically the numerical performance

## Geometrically exact beam – Kinematical assumptions

- Reference and current configuration



- Restricted position field

$$\mathbf{x}(\theta^\alpha, s, t) = \varphi(s,t) + \theta^\alpha \mathbf{d}_\alpha(s,t)$$

where $\mathbf{R}(s,t) \in SO(3)$ such that

$$\mathbf{d}_k(s) = \mathbf{R}(s,t)\mathbf{e}_k\,, \quad \text{with } \mathbf{R} = \mathbf{d}_k \otimes \mathbf{e}^k$$

- Skew-symmetric effective curvature $\tilde{\mathbf{k}}$

$$(\mathbf{d}_k)_{,s} = \tilde{\mathbf{k}}\mathbf{d}_k = \mathbf{k} \times \mathbf{d}_k\,, \quad \text{with } \tilde{\mathbf{k}} = \mathbf{R}_{,s}\mathbf{R}^{-1} = \mathbf{d}_{i,s} \otimes \mathbf{d}^i$$

## Virtual work and director formulation

- Virtual work

$$\delta W = \int_{s_1}^{s_2} \left\{ \delta\varphi \cdot (A_\rho \ddot{\varphi} + \ddot{\mathbf{q}} - \overline{\mathbf{n}} - \mathbf{n}_{,s}) + \delta\phi \cdot \left(\mathbf{q} \times \ddot{\varphi} + \dot{\mathbf{h}} - \overline{\mathbf{m}} - \mathbf{m}_{,s} - \varphi_{,s} \times \mathbf{n}\right) \right\} \mathrm{d}s$$
$$+ \left(\delta\varphi \cdot (\mathbf{n} - \overline{\mathbf{n}}) + \delta\phi \cdot (\overline{\mathbf{m}} - \mathbf{m})\right)\Big|_0^L = 0 \quad \forall \delta\varphi, \delta\phi, t$$

- Constitutive law

$$W(\gamma_i, k^i) = \frac{1}{2} d^{1/2} \gamma_i (\hat{\mathbf{D}}_1)^{ij} \gamma_j + \frac{1}{2} d^{1/2} k^i (\hat{\mathbf{D}}_2)_{ij} k^j$$

with

$$[\hat{\mathbf{D}}_1] = \mathrm{Diag}[GA_1, GA_2, EA] \quad \text{and} \quad [\hat{\mathbf{D}}_2] = \mathrm{Diag}[EI_1, EI_2, GJ]$$

- Variational formulation of the geometrically exact beam

$$\delta W = \int_{s_1}^{s_2} \Big\{ \delta\varphi \cdot \left[A_\rho \ddot{\varphi} + q_\rho^\beta \ddot{\mathbf{d}}_\beta - \overline{\mathbf{n}}\right] + \delta\mathbf{d}_\alpha \cdot \left[M_\rho^{\alpha\beta} \ddot{\mathbf{d}}_\beta + q_\rho^\alpha \ddot{\varphi}\right] - \delta\mathbf{d}_i \cdot \frac{1}{2}(\mathbf{d}^i \times \overline{\mathbf{m}})$$
$$+ \lambda^{ij} \delta\mathbf{d}_i \cdot \mathbf{d}_j + \mathbf{n} \cdot \delta\varphi_{,s} + \frac{1}{2}\left[(\mathbf{n} \cdot \mathbf{d}^i)(\varphi_{,s} \cdot \delta\mathbf{d}_i) - (\delta\mathbf{d}_i \cdot \mathbf{n})(\mathbf{d}^i \cdot \varphi_{,s})\right]$$
$$+ \frac{1}{2}(\mathbf{m} \cdot \mathbf{d}_i)\varepsilon_{ijk} \left[\delta(d^{-1/2})(\mathbf{d}_k \cdot \mathbf{d}_{j,s}) + d^{-1/2}\delta\mathbf{d}_k \cdot \mathbf{d}_{j,s} + d^{-1/2}(\mathbf{d}_k \cdot \delta\mathbf{d}_{j,s})\right] \Big\} \mathrm{d}s$$
$$- \left(\delta\varphi \cdot \overline{\mathbf{n}} + \delta\mathbf{d}_i \cdot \frac{1}{2}(\mathbf{d}^i \times \overline{\mathbf{m}})\right)\Big|_{s_1}^{s_2} = 0 \quad \forall \delta\varphi, \delta\mathbf{d}_i, t$$

## Finite element formulation

- Galerkin type approach

$$\varphi^{\mathrm{h}}(s,t) = \sum_A N^A(s)\varphi_A(t), \quad \mathbf{d}_i^{\mathrm{h}}(s,t) = \sum_A N^A(s)\mathbf{d}_{iA}(t)$$
$$\delta\varphi^{\mathrm{h}}(s) = \sum_A N^A(s)\delta\varphi_A, \quad \delta\mathbf{d}_i^{\mathrm{h}}(s) = \sum_A N^A(s)\delta\mathbf{d}_{iA}$$

- Virtual work of the contact force

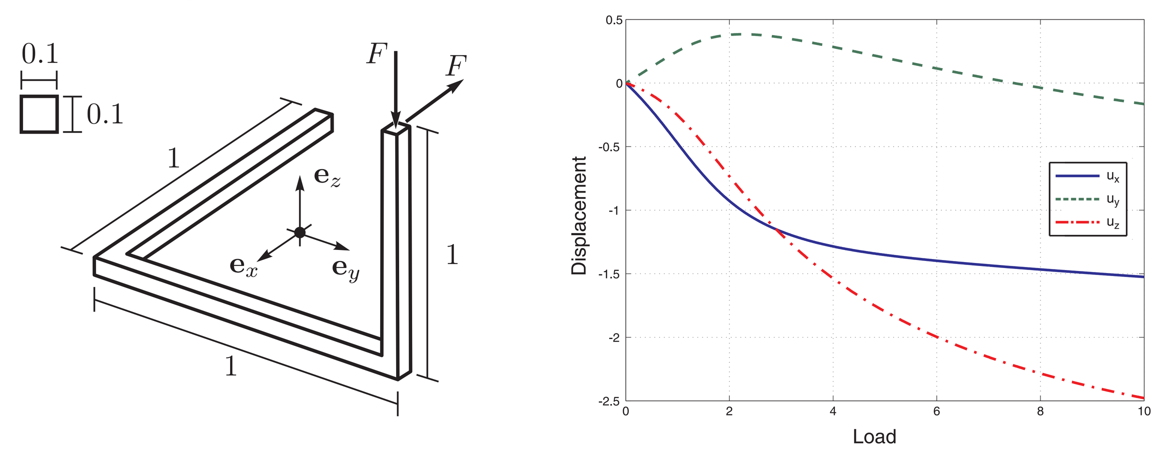
$$\delta W_1^{\mathrm{int,h}} = \delta\varphi_A \cdot \int_{s_1}^{s_2} N_{,s}^A \mathbf{n}^{\mathrm{h}} \,\mathrm{d}s +$$
$$\sum_{i=1}^{3} \delta\mathbf{d}_{iA} \cdot \int_{s_1}^{s_2} \frac{1}{2}\left[(\mathbf{n}^{\mathrm{h}} \cdot \mathbf{d}^{i,\mathrm{h}}) N^A N_{,s}^B \varphi_B - (\varphi_{,s}^h \cdot \mathbf{d}^{i,\mathrm{h}}) N^A \mathbf{n}^{\mathrm{h}}\right] \mathrm{d}s$$

- Virtual work of the contact torques

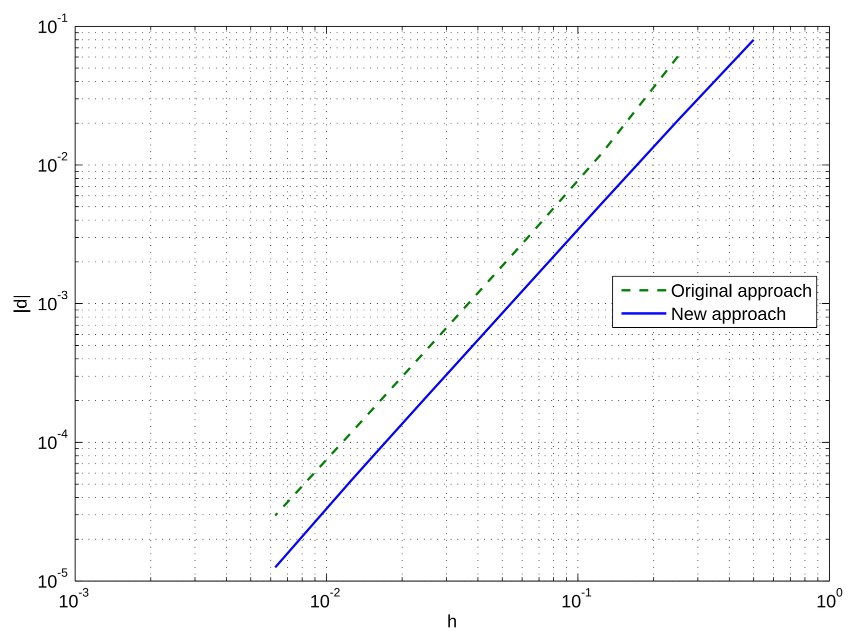
$$\delta W_2^{\mathrm{int,h}} = \frac{1}{2}\delta\mathbf{d}_{iA} \cdot \int_{s_1}^{s_2} \det(\mathbf{R}^{-1,\mathrm{h}})(\mathbf{m}^{\mathrm{h}} \cdot \mathbf{d}_k^{\mathrm{h}})\varepsilon_{kji}(N^A N_{,s}^B \mathbf{d}_{jB} - N_{,s}^A N^B \mathbf{d}_{jB})\,\mathrm{d}s -$$
$$\frac{1}{2}\delta\mathbf{d}_{1A} \cdot \int_{s_1}^{s_2} (\det(\mathbf{R}^{-1,\mathrm{h}}))^2 N^A(\mathbf{d}_2^{\mathrm{h}} \times \mathbf{d}_3^{\mathrm{h}})(\mathbf{m}^{\mathrm{h}} \cdot \mathbf{d}_l^{\mathrm{h}})\varepsilon_{lmn}(\mathbf{d}_n^{\mathrm{h}} \cdot \mathbf{d}_{m,s}^{\mathrm{h}})\,\mathrm{d}s -$$
$$\frac{1}{2}\delta\mathbf{d}_{2A} \cdot \int_{s_1}^{s_2} (\det(\mathbf{R}^{-1,\mathrm{h}}))^2 N^A(\mathbf{d}_3^{\mathrm{h}} \times \mathbf{d}_1^{\mathrm{h}})(\mathbf{m}^{\mathrm{h}} \cdot \mathbf{d}_l^{\mathrm{h}})\varepsilon_{lmn}(\mathbf{d}_n^{\mathrm{h}} \cdot \mathbf{d}_{m,s}^{\mathrm{h}})\,\mathrm{d}s -$$
$$\frac{1}{2}\delta\mathbf{d}_{3A} \cdot \int_{s_1}^{s_2} (\det(\mathbf{R}^{-1,\mathrm{h}}))^2 N^A(\mathbf{d}_1^{\mathrm{h}} \times \mathbf{d}_2^{\mathrm{h}})(\mathbf{m}^{\mathrm{h}} \cdot \mathbf{d}_l^{\mathrm{h}})\varepsilon_{lmn}(\mathbf{d}_n^{\mathrm{h}} \cdot \mathbf{d}_{m,s}^{\mathrm{h}})\,\mathrm{d}s$$

## Numerical example

Beam with slope discontinuity




Geometry of the structure and tip displacement versus load $F$.




Errors in tip displacements versus h-refinement.

### References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  Frame-indifferent beam finite elements based upon the geometrically exact beam theory.
  *Int. J. Numer. Meth. Engng*, 54:1775–1788, 2002.
- S.R. Eugster, C. Hesch, P. Betsch and Ch. Glocker
  Director-based beam finite elements relying on the geometrically exact beam theory formulated in skew coordinates
  *Int. J. Numer. Meth. Engng*, 97:111–129, 2014