Karlsruhe Institute of Technology

UNIVERSITÄT SIEGEN

Finite deformation phase-field approach to fracture

C. Hesch and K. Weinberg

Introduction

Fracture mechanics [1]

- Classical brittle fracture approaches of Griffith and Irwin
- Material fails locally upon a specific fracture energy
- Continuum formulation - solve via standard finite-element strategies

Thermodynamically consistent approach [2]

- Finite deformation approach - multiplicative decomposition of the local deformation
- Enables the use of phase-field models for general nonlinear constitutive laws
- Energy-momentum consistent time integration scheme

Field equations

- Phase field equation

$$\mathfrak{s} - l^2\Delta\mathfrak{s} = 0 \text{ in } \mathcal{B}_0$$
$$\nabla\mathfrak{s}\cdot\boldsymbol{n} = 0 \text{ on } \delta\mathcal{B}_0$$

- Regularized crack surface topology

$$\Gamma_l(\mathfrak{s}) := \int_{\mathcal{B}_0} \gamma_l(\mathfrak{s})\,\mathrm{d}V = \int_{\mathcal{B}_0} \frac{1}{2l}\mathfrak{s}^2 + \frac{l}{2}\nabla(\mathfrak{s})\cdot\nabla(\mathfrak{s})\,\mathrm{d}V$$

- Local balance of linear momentum

$$\dot{\varphi} = \boldsymbol{v},\ \rho_0\dot{\boldsymbol{v}} = \mathrm{Div}\,\boldsymbol{P} + \bar{\boldsymbol{B}}$$
$$\varphi = \bar{\varphi} \text{ on } \partial\mathcal{B}_0^{\varphi}\times\mathcal{I},\ \boldsymbol{PN} = \bar{\boldsymbol{T}} \text{ on } \partial\mathcal{B}_0^{\sigma}\times\mathcal{I}$$

- Weak form of the nonlinear problem

$$\int_{\mathcal{B}_0}\delta\varphi\cdot\rho_0\dot{\boldsymbol{v}}\,\mathrm{d}V + \int_{\mathcal{B}_0}\boldsymbol{S}:\boldsymbol{F}^T\nabla_{\boldsymbol{X}}(\delta\varphi)\,\mathrm{d}V = \int_{\mathcal{B}_0}\delta\varphi\cdot\bar{\boldsymbol{B}}\,\mathrm{d}V + \int_{\partial\mathcal{B}_0^{\sigma}}\delta\varphi\cdot\bar{\boldsymbol{T}}\,\mathrm{d}A$$

Operator split

- Decomposition into compressive and tensile part

$$\boldsymbol{F} = \boldsymbol{F}^-\boldsymbol{F}^+ = \sum_{a=1}^{n}\lambda_a^-\lambda_a^+\boldsymbol{n}_a\otimes\boldsymbol{N}_a$$

- Further decomposition of tensile stretches

$$\boldsymbol{F} = \sum_{a=1}^{n}(\lambda_a^+)^{\mathfrak{s}}(\lambda_a^+)^{(1-\mathfrak{s})}\lambda_a^-\boldsymbol{n}_a\otimes\boldsymbol{N}_a$$

- Elastic, fracture insensitive part

$$\boldsymbol{F}^{\mathrm{e}} = \sum_{a=1}^{n}(\lambda_a^+)^{(1-\mathfrak{s})}\lambda_a^-\boldsymbol{n}_a\otimes\boldsymbol{N}_a,\ \boldsymbol{C}^{\mathrm{e}} = (\boldsymbol{F}^{\mathrm{e}})^T\boldsymbol{F}^{\mathrm{e}}$$

- Corresponding second Piola–Kirchhoff stress tensor

$$\boldsymbol{S}^{\mathrm{e}} = 2\frac{\partial\Psi(\boldsymbol{C}^{\mathrm{e}},\mathfrak{s})}{\partial\boldsymbol{C}} = \sum_{a=1}^{n}\frac{1}{\lambda_a}\frac{\partial\Psi}{\partial\lambda_a^{\mathrm{e}}}\frac{\partial\lambda_a^{\mathrm{e}}}{\partial\lambda_a}\boldsymbol{N}_a\otimes\boldsymbol{N}_a$$

- Link to linear theory: $\epsilon_a = \log(\lambda_a)$

$$\epsilon^{\mathrm{e}} = \sum_{a=1}^{n}\left((1-\mathfrak{s})\epsilon_a^+\boldsymbol{N}_a\otimes\boldsymbol{N}_a + \epsilon_a^-\boldsymbol{N}_a\otimes\boldsymbol{N}_a\right)$$

Consistent discretisation

- Conform discretisation of the fields

$$\varphi^h = \sum_{A\in\omega}N^A(\boldsymbol{X})\boldsymbol{q}_A,\quad \delta\varphi^h = \sum_{A\in\omega}N^A(\boldsymbol{X})\delta\boldsymbol{q}_A$$
$$\mathfrak{s}^h = \sum_{A\in\omega}N^A(\boldsymbol{X})\mathfrak{s}_A,\quad \delta\mathfrak{s}^h = \sum_{A\in\omega}N^A(\boldsymbol{X})\delta\mathfrak{s}_A$$

- Discrete phase field

$$\delta\mathfrak{s}_A\left[\int_{\mathcal{B}_0}\frac{g_c}{l}N^AN^B\mathfrak{s}_B + g_cl\nabla N^A\cdot\nabla N^B\mathfrak{s}_B + \frac{\partial\Psi}{\partial\mathfrak{s}_A}\,\mathrm{d}V\right] = 0$$

- Full discrete system

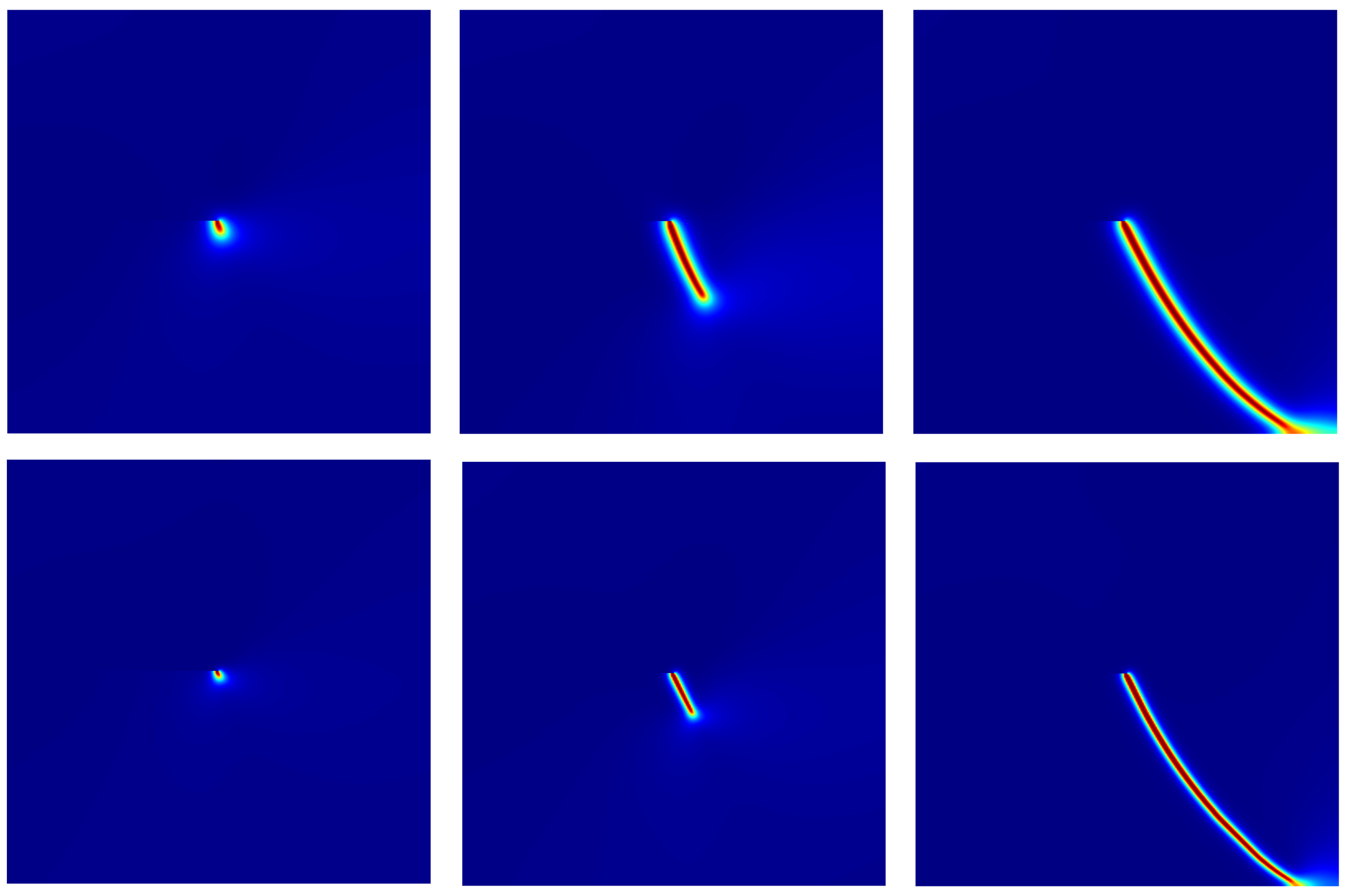
$$\delta\boldsymbol{q}_A\cdot\left[M^{AB}\frac{\boldsymbol{v}_{B,n+1}-\boldsymbol{v}_{B,n}}{\Delta t} + \int_{\mathcal{B}_0}\nabla N^A(\boldsymbol{X})\cdot\boldsymbol{S}^{\mathrm{e},h}_{n,n+1}\nabla N^B(\boldsymbol{X})\,\mathrm{d}V\boldsymbol{q}_{B,n+1/2}\right] = \delta\boldsymbol{q}_A\cdot\left[\boldsymbol{F}^{A,ext}_{n+1/2}\right]$$
$$\delta\mathfrak{s}_A\left[\int_{\mathcal{B}_0}\frac{g_c}{l}N^AN^B\mathfrak{s}_{B,n+1/2} + g_cl\nabla N^A\cdot\nabla N^B\mathfrak{s}_{B,n+1/2}\,\mathrm{d}V\right] = -\delta\mathfrak{s}_A\int_{\mathcal{B}_0}\left(\frac{\partial\Psi}{\partial\mathfrak{s}_A}\right)_{n+1/2}\mathrm{d}V$$

- Discrete gradient

$$\boldsymbol{S}^{\mathrm{e},h}_{n,n+1} = 2\frac{\partial\Psi_{n+1/2}}{\partial\boldsymbol{C}^h} + 2\frac{\Psi_{n+1}-\Psi_n+g_c(\gamma_{n+1}-\gamma_n)-\frac{\partial\Psi_{n+1/2}}{\partial\boldsymbol{C}^h}:\Delta\boldsymbol{C}^h}{\Delta\boldsymbol{C}^h:\Delta\boldsymbol{C}^h}\Delta\boldsymbol{C}^h$$

Numerical example

Pure shear test - uniform mesh with 256×256 elements



Phase-field patterns for non-linear shear test. Upper row at a displacement of $u = [9.0, 11.0, 13.3]\times10^{-3}$ mm for a length scale of 0.0150 mm. Lower row at a displacement of $u = [9.0, 11.0, 15.1]\times10^{-6}$ mm for a length scale of 0.0075 mm.

References

- C. Miehe, M. Hofacker and F. Welschinger
  Thermodynamically consistent phase-field models of fracture: Variational principles and multi-field FE implementations.
  Int. J. Numer. Meth. Engng, 83:1273–1311, 2010.
- C. Hesch and K. Weinberg
  Thermodynamically consistent algorithms for a finite-deformation phase-field approach to fracture
  Int. J. Numer. Meth. Engng, DOI: 10.1002/nme.4709