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Design of an energy and momentum consistent time integration scheme based on a polyconvex inspired mixed thermo-electro-mechanic framework

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Motivation

Dielectric elastomers are being considered for a wide variety of applications, particularly artificial muscles. In order to predict the behavior of these electronic components as accurately as possible, it is essential to develop suitable simulation programs that take into account the thermo-electro-mechanical processes occurring within them.

In this work, a novel mixed formulation for these kind of coupled problems is derived, which is both energy and momentum consistent.

Design of the novel mixed framework

Fundamental equations

- Tensor cross product

$$(\mathbf{A} \mathbin{\times\!\!\!\!\times} \mathbf{B})_{ij} = \varepsilon_{i\alpha\beta}\varepsilon_{jab}A_{\alpha a}B_{\beta b}$$

- Kinematics [1]

$$\mathbf{C}_\varphi = \mathbf{F}_\varphi^{\mathrm{T}}\mathbf{F}_\varphi\,, \qquad \mathbf{G}_\varphi = \frac{1}{2}\mathbf{C}_\varphi \mathbin{\times\!\!\!\!\times} \mathbf{C}_\varphi\,, \qquad C_\varphi = \frac{1}{3}\mathbf{G}_\varphi : \mathbf{C}_\varphi$$

- Momentum balance

$$\rho_0\dot{\mathbf{v}} - \mathrm{Div}(\mathbf{F}_\varphi \mathbf{S}_\varphi) - \bar{\mathbf{B}} = \mathbf{0}$$

- Energy balance [2]

$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta\eta) - \dot{\theta}\eta + \mathrm{Div}\,\mathbf{Q} - \bar{R} = 0$$

- Gauss's and Faraday's law [2]

$$\mathrm{Div}\,\mathbf{D}_0 - \rho_0^e = 0, \qquad \mathbf{E}_0 = -\nabla_{\mathbf{X}}\Phi$$

Polyconvex inspired internal energy function [2]

$\widehat{W}(\mathbf{C},\,\mathbf{G},\,C,\,\mathbf{D}_0,\,\theta) \longrightarrow$ certain convexity criteria apply.

Constitutive equations [2]

$$\eta = -\partial_\theta\widehat{W}, \qquad \mathbf{E}_0 = \partial_{\mathbf{D}_0}\widehat{W}, \qquad \mathbf{Q} = -k_0 C^{-1}\mathbf{G}\nabla_{\mathbf{X}}\theta$$

Equivalent mixed formulation of the momentum balance

$$\begin{aligned}
\rho_0\dot{\mathbf{v}} - \mathrm{Div}\left(2\mathbf{F}_\varphi\boldsymbol{\Lambda}^{\mathbf{C}}\right) - \bar{\mathbf{B}} &= \mathbf{0}\\
\partial_{\mathbf{C}}\widehat{W} - \boldsymbol{\Lambda}^{\mathbf{C}} + \boldsymbol{\Lambda}^{\mathbf{G}} \mathbin{\times\!\!\!\!\times} \mathbf{C} + \frac{1}{3}\Lambda^C\mathbf{G} &= \mathbf{0}\\
\partial_{\mathbf{G}}\widehat{W} - \boldsymbol{\Lambda}^{\mathbf{G}} + \frac{1}{3}\Lambda^C\mathbf{C} &= \mathbf{0}\\
\partial_C\widehat{W} - \Lambda^C &= 0\\
\mathbf{F}_\varphi^{\mathrm{T}}\mathbf{F}_\varphi - \mathbf{C} &= \mathbf{0}\\
\frac{1}{2}\mathbf{C} \mathbin{\times\!\!\!\!\times} \mathbf{C} - \mathbf{G} &= \mathbf{0}\\
\frac{1}{3}\mathbf{G} : \mathbf{C} - C &= 0
\end{aligned}$$

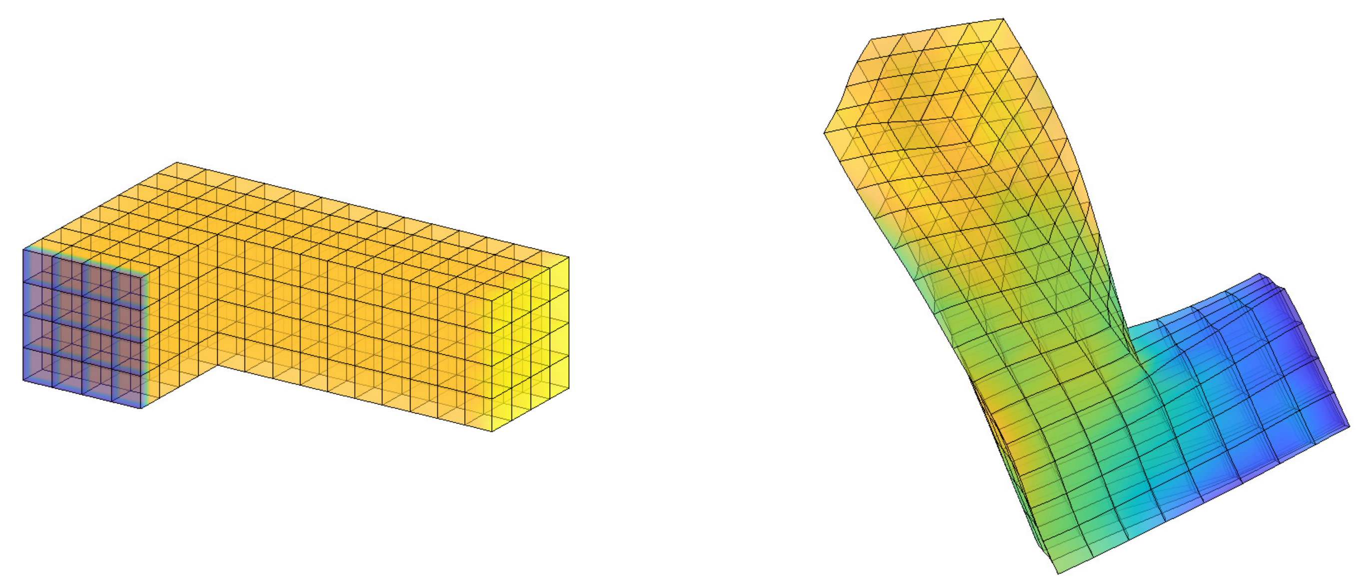
$\longrightarrow$ In the mixed framework the kinematic quantities $\mathbf{C}$, $\mathbf{G}$ and C as well as the Lagrange multipliers $\boldsymbol{\Lambda}^{\mathbf{C}}$, $\boldsymbol{\Lambda}^{\mathbf{G}}$ and Λ^C are considered to be independent variables.

Discrete derivatives [3]

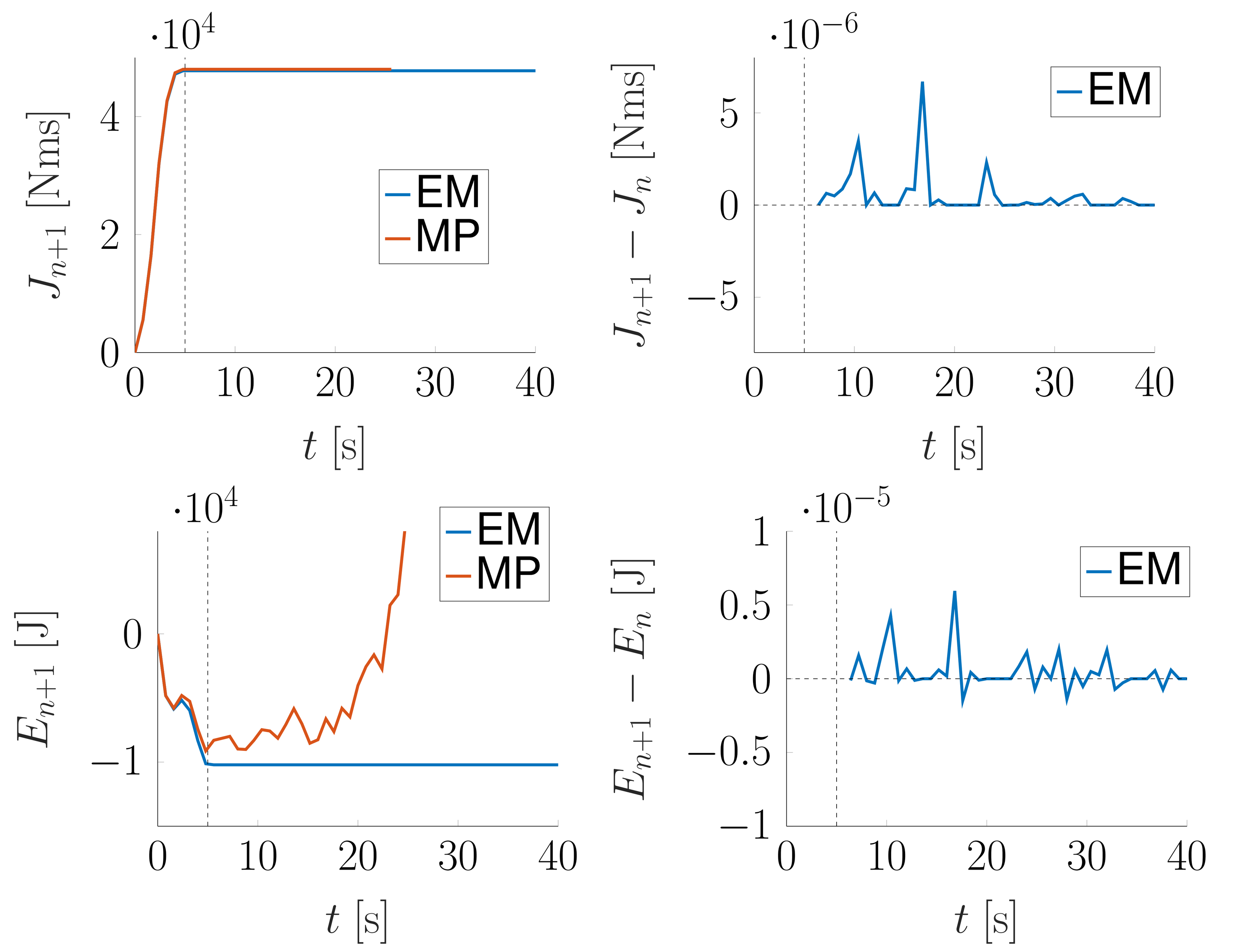
- Algorithmic or time-discrete counterparts of $\partial_{\mathbf{C}}\widehat{W}$, $\partial_{\mathbf{G}}\widehat{W}$, $\partial_C\widehat{W}$, $\partial_{\mathbf{D}_0}\widehat{W}$ and $\partial_\theta\widehat{W}$.
- The so-called directionality property of the discrete derivatives is substantially responsible for the energy consistency of the formulation. In combination with a properly chosen time discretization of the underlying equations, they lead to an energy-momentum scheme.

Numerical example: Flying L-shaped block

- An L-shaped block is subjected to various external loads during $t \leq 5\,\mathrm{s}$. Afterwards, the body tumbles freely through space.
- Snapshots of the temperature distribution at $t = 0\,\mathrm{s}$ and $t = 40\,\mathrm{s}$



- Evolution of the total angular momentum and the total energy for the energy-momentum scheme (EM) and the midpoint rule (MP)



References

[1] Betsch, P., Janz, A., and Hesch, C. A mixed variational framework for the design of energy-momentum schemes inspired by the structure of polyconvex stored energy functions. In: *Computer Methods in Applied Mechanics and Engineering*, 335: 660–696, June 2018.

[2] Franke, M., Ortigosa, R., Martínez-Frutos, J., Gil, A., and Betsch, P. A thermodynamically consistent time integration scheme for non-linear thermo-electro-mechanics. In: *Computer Methods in Applied Mechanics and Engineering*, 389: 114298, Feb. 2022.

[3] Gonzalez, O. Time integration and discrete Hamiltonian systems. In: *Journal of Nonlinear Science*, 6(5): 449–467, Sept. 1996.