Energy-momentum schemes for large deformation contact problems

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Dynamic contact problems in elasticity are dealt with in the framework of nonlinear finite element methods. A new energy-momentum conserving time-stepping scheme for the mortar contact formulation is presented. The proposed method relies on a reparametrisation of the contact constraints in terms of specific invariants. For the time discretisation of the contact forces emanating from the mortar formulation the notion of a discrete gradient is applied.

1 Hamiltonian formulation of discrete elastodynamics

We start with the space finite element discretisation of nonlinear elastodynamics, which gives rise to a discrete strain energy function, given by

\begin{equation}
V^{int}(\mathbf{q}) = \int_B W(\mathbf{C}) \, d\mathbf{V}
\end{equation}

where \( \mathbf{q} \) represents a possible discrete configuration and \( \mathbf{C} \) the discrete version of the deformation tensor (right Cauchy-Green tensor). Furthermore, we assume, that the external forces can be derived from an energy potential

\begin{equation}
V^{ext}(\mathbf{q}) = -\int_B \rho_0 \mathbf{b} \cdot \varphi \, d\mathbf{V} - \int_{\partial B_{ext}} \mathbf{T} \cdot \varphi \, d\mathbf{A}
\end{equation}

where \( \rho_0 \) denotes the reference mass density, \( \mathbf{b} \) the applied body force, \( \varphi \) the actual configuration and \( \mathbf{T} \) the prescribed traction boundary condition. Due to the presence of contact constraints, the equations of motion pertaining to the fully discrete system can be written by using a mid-point-type discretisation of the underlying system of differential algebraic equations:

\begin{equation}
\mathbf{q}_{n+1} - \mathbf{q}_n = \frac{\Delta t}{2}(\mathbf{v}_n + \mathbf{v}_{n+1})
\end{equation}

\begin{equation}
\mathbf{M}(\mathbf{v}_{n+1} - \mathbf{v}_n) = -\Delta t \nabla \Phi^T_\pi \mathbf{V}^{(\mathbf{q}_n, \mathbf{q}_{n+1})} - \Delta t \sum_{l=1}^m \lambda_l n \nabla \Phi_l(\mathbf{q}_n, \mathbf{q}_{n+1})
\end{equation}

Here, the discrete gradient is defined as

\begin{equation}
\nabla \mathbf{V}^{(\mathbf{q}_n, \mathbf{q}_{n+1})} = D \pi(\mathbf{q}_{n+q})^T \nabla \mathbf{V}(\pi(\mathbf{q}_n), \pi(\mathbf{q}_{n+1}))
\end{equation}

with

\begin{equation}
\nabla \mathbf{V}(\pi(\mathbf{q}_n), \pi(\mathbf{q}_{n+1})) = \nabla \mathbf{V}(\pi_{n+\frac{1}{2}}) + \frac{V(\pi_{n+1}) - V(\pi_n) - \nabla \mathbf{V}(\pi_{n+\frac{1}{2}}) \cdot (\pi_{n+1} - \pi_n)}{\|\pi_{n+1} - \pi_n\|^2}(\pi_{n+1} - \pi_n)
\end{equation}

using a reparametrisation with possible invariants \( \pi \), which have to be members of either the set \( \mathcal{S} \) or \( \mathcal{T} \)

\begin{equation}
\mathcal{S}(\mathbf{q}) = \{ \mathbf{q}_A \cdot \mathbf{q}_B, \, 1 \leq A \leq B \leq n_{\text{nodes}} \}
\end{equation}

\begin{equation}
\mathcal{T}(\mathbf{q}) = \{ \det([\mathbf{q}_A \cdot \mathbf{q}_B]), \, 1 \leq A \leq B \leq n_{\text{nodes}} \}
\end{equation}

Possible invariants of the strain energy function can be identified as the components of the deformation tensor \( \mathbf{C} \). The same approach can be applied to the contact constraint functions, as shown next.

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2 Mortar method

The reparametrisation of the mortar constraints can be carried out with at least five invariants, three out of $S$ and two out of $T$:

\begin{align*}
\pi_1(q_{seg}) &= (x_2^{(1)} - x_1^{(1)}) \cdot (x_2^{(1)} - x_1^{(1)}) \\
\pi_2(q_{seg}) &= (x_2^{(1)} - x_1^{(1)}) \cdot (x_2^{(2)} - x_1^{(1)}) \\
\pi_3(q_{seg}) &= (x_2^{(1)} - x_1^{(1)}) \cdot (x_2^{(2)} - x_1^{(1)}) \\
\pi_4(q_{seg}) &= (x_2^{(1)} - x_1^{(1)}) \cdot \Lambda (-2x_1^{(1)} + x_2^{(1)} + x_2^{(2)}) \\
\pi_5(q_{seg}) &= (x_2^{(1)} - x_1^{(1)}) \cdot \Lambda (x_2^{(2)} - x_2^{(2)})
\end{align*}

(7)

where $x_A^{(B)}$ denotes the four element nodes, defining a mortar segment (see Betsch & Hesch [1]) and $\Lambda$ is a constant skew-symmetric matrix with

\begin{equation}
\Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\end{equation}

A straightforward calculation shows, that the contact constraint functions can be recast as

\begin{align*}
\Phi_1^\text{seq}(\pi(q_{seg})) &= \frac{1}{16} (\xi_b^{(1)} - \xi_a^{(1)}) \left\{ \pi_4 \int_{-1}^{1} (\xi^{(1)} - 1) \, d\eta + \pi_5 \int_{-1}^{1} (\xi^{(2)} - \xi^{(1)} \xi^{(2)}) \, d\eta \right\} \\
\Phi_2^\text{seq}(\pi(q_{seg})) &= \frac{1}{16} (\xi_b^{(1)} - \xi_a^{(1)}) \left\{ \pi_5 \int_{-1}^{1} (\xi^{(2)} + \xi^{(1)} \xi^{(2)}) \, d\eta - \pi_4 \int_{-1}^{1} (\xi^{(1)} + 1) \, d\eta \right\}
\end{align*}

(9)

and

(10)

again defined for one specific mortar segment, restricted by the local parametrisation $\xi$. A similar approach can be applied to the node-to-segment contact formulation (see Betsch & Hesch [2]).

3 Numerical example

The numerical example deals with the planar model of a bearing depicted in Fig. 1. The bearing consists of two rings (Youngs’s modulus $E = 10^5$, Poissons’s ratio $\nu = 0.1$ and mass density $\varrho_R = 0.001$), which are discretized by 4-node isoparametric displacement-based plain strain elements. The discretization of the outer ring relies on 10x48 elements, for the inner ring 10x40 have been used.

For $t \in [0, 0.5]$, a torque acts on the inner ring in form of a hat function over time. Then, for $t \in (0.5, 2]$, no external loads are acting on the bearing anymore. Fig. 2 shows that for $t \geq 0.5$ the present scheme does indeed conserve the total energy for the frictionless contact problem under consideration.

![Fig. 1: Discretized bearing](image1)

![Fig. 2: Energy versus time](image2)

Literatur
