Energy-momentum schemes for large deformation contact problems

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Dynamic contact problems in elasticity are dealt with in the framework of nonlinear finite element methods. A new energy-momentum conserving time-stepping scheme for the mortar contact formulation is presented. The proposed method relies on a reparametrization of the contact constraints in terms of specific invariants. For the time discretisation of the contact forces emanating from the mortar formulation the notion of a discrete gradient is applied.

1 Hamiltonian formulation of discrete elastodynamics

We start with the space finite element discretisation of nonlinear elastodynamics, which gives rise to a discrete strain energy function, given by

$$V^{\text{int}}(\mathbf{q}) = \int_{\mathcal{B}} W(\mathbf{C})\,\mathrm{d}V \tag{1}$$

where $\mathbf{q}$ represents a possible discrete configuration and $\mathbf{C}$ the discrete version of the deformation tensor (right Cauchy-Green tensor). Furthermore, we assume, that the external forces can be derived from an energy potential

$$V^{\text{ext}}(\mathbf{q}) = -\int_{\mathcal{B}} \rho_0 \mathbf{b}\cdot\boldsymbol{\varphi}\,\mathrm{d}V - \int_{\partial\mathcal{B}_\sigma} \bar{\mathbf{t}}\cdot\boldsymbol{\varphi}\,\mathrm{d}A \tag{2}$$

where ρ_0 denotes the reference mass density, $\mathbf{b}$ the applied body force, $\boldsymbol{\varphi}$ the actual configuration and $\bar{\mathbf{t}}$ the prescribed traction boundary condition. Due to the presence of contact constraints, the equations of motion pertaining to the fully discrete system can be written by using a mid-point-type discretisation of the underlying system of differential algebraic equations:

$$\begin{aligned} \mathbf{q}_{n+1} - \mathbf{q}_n &= \frac{\Delta t}{2}(\mathbf{v}_n + \mathbf{v}_{n+1}) \\ \mathbf{M}(\mathbf{v}_{n+1} - \mathbf{v}_n) &= -\Delta t \overline{\nabla}_{\mathrm{q}} V(\mathbf{q}_n, \mathbf{q}_{n+1}) - \Delta t \sum_{l=1}^{m} (\lambda_l)_{n+1} \overline{\nabla}_{\mathrm{q}} \Phi_l(\mathbf{q}_n, \mathbf{q}_{n+1}) \\ \mathbf{0} &= \boldsymbol{\Phi}(\mathbf{q}_{n+1}) \end{aligned} \tag{3}$$

Here, the discrete gradient is defined as

$$\overline{\nabla}_{\mathrm{q}} V(\mathbf{q}_n, \mathbf{q}_{n+1}) = \mathrm{D}\,\boldsymbol{\pi}(\mathbf{q}_{n+q})^T \overline{\overline{\nabla}}_{\mathrm{q}} V(\boldsymbol{\pi}(\mathbf{q}_n), \boldsymbol{\pi}(\mathbf{q}_{n+1})) \tag{4}$$

with

$$\overline{\overline{\nabla}}_{\mathrm{q}} V(\boldsymbol{\pi}(\mathbf{q}_n), \boldsymbol{\pi}(\mathbf{q}_{n+1})) = \nabla_{\pi} V(\boldsymbol{\pi}_{n+\frac{1}{2}}) + \frac{V(\boldsymbol{\pi}_{n+1}) - V(\boldsymbol{\pi}_n) - \nabla_{\pi} V(\boldsymbol{\pi}_{n+\frac{1}{2}})\cdot(\boldsymbol{\pi}_{n+1} - \boldsymbol{\pi}_n)}{\|\boldsymbol{\pi}_{n+1} - \boldsymbol{\pi}_n\|^2}(\boldsymbol{\pi}_{n+1} - \boldsymbol{\pi}_n) \tag{5}$$

using a reparametrisation with possible invariants $\boldsymbol{\pi}$, which have to be members of either the set $\mathbb{S}$ or $\mathbb{T}$

$$\begin{aligned} \mathbb{S}(\mathbf{q}) &= \{\mathbf{q}_A \cdot \mathbf{q}_B,\ 1 \le A \le B \le n_{\text{nodes}}\} \\ \mathbb{T}(\mathbf{q}) &= \{\det([\mathbf{q}_A, \mathbf{q}_B]),\ 1 \le A \le B \le n_{\text{nodes}}\} \end{aligned} \tag{6}$$

Possible invariants of the strain energy function can be identified as the components of the deformation tensor $\mathbf{C}$. The same approach can be applied to the contact constraint functions, as shown next.

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2 Mortar method

The reparametrisation of the mortar constraints can be carried out with at least five invariants, three out of $\mathbb{S}$ and two out of $\mathbb{T}$

$$\begin{aligned}
\pi_1(\mathbf{q}_{\text{seg}}) &= (\mathbf{x}_2^{(1)} - \mathbf{x}_1^{(1)}) \cdot (\mathbf{x}_2^{(1)} - \mathbf{x}_1^{(1)}) \\
\pi_2(\mathbf{q}_{\text{seg}}) &= (\mathbf{x}_2^{(1)} - \mathbf{x}_1^{(1)}) \cdot (\mathbf{x}_1^{(2)} - \mathbf{x}_1^{(1)}) \\
\pi_3(\mathbf{q}_{\text{seg}}) &= (\mathbf{x}_2^{(1)} - \mathbf{x}_1^{(1)}) \cdot (\mathbf{x}_2^{(2)} - \mathbf{x}_1^{(1)}) \\
\pi_4(\mathbf{q}_{\text{seg}}) &= (\mathbf{x}_2^{(1)} - \mathbf{x}_1^{(1)}) \cdot \mathbf{\Lambda}(-2\mathbf{x}_1^{(1)} + \mathbf{x}_1^{(2)} + \mathbf{x}_2^{(2)}) \\
\pi_5(\mathbf{q}_{\text{seg}}) &= (\mathbf{x}_2^{(1)} - \mathbf{x}_1^{(1)}) \cdot \mathbf{\Lambda}(\mathbf{x}_1^{(2)} - \mathbf{x}_2^{(2)})
\end{aligned} \tag{7}$$

where $\mathbf{x}_A^{(B)}$ denotes the four element nodes, defining a mortar segment (see Betsch & Hesch [1]) and $\mathbf{\Lambda}$ is a constant skew-symmetric matrix with

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{8}$$

A straightforward calculation shows, that the contact constraint functions can be recast as

$$\boxed{\Phi_1^{seq}(\boldsymbol{\pi}(\mathbf{q}_{seg})) = \frac{1}{16}(\xi_b^{(1)} - \xi_a^{(1)}) \left\{ \pi_4 \int_{-1}^{1} (\xi^{(1)} - 1)\, d\eta + \pi_5 \int_{-1}^{1} (\xi^{(2)} - \xi^{(1)}\xi^{(2)})\, d\eta \right\}} \tag{9}$$

and

$$\boxed{\Phi_2^{seq}(\boldsymbol{\pi}(\mathbf{q}_{seg})) = \frac{1}{16}(\xi_b^{(1)} - \xi_a^{(1)}) \left\{ \pi_5 \int_{-1}^{1} (\xi^{(2)} + \xi^{(1)}\xi^{(2)})\, d\eta - \pi_4 \int_{-1}^{1} (\xi^{(1)} + 1)\, d\eta \right\}} \tag{10}$$

again defined for one specific mortar segment, restricted by the local parametrisation ξ. A similar approach can be applied to the node-to-segment contact formulation (see Betsch & Hesch [2]).

3 Numerical example

The numerical example deals with the planar model of a bearing depicted in Fig. 1. The bearing consists of two rings (Youngs's modulus $E = 10^5$, Poissons's ratio $\nu = 0.1$ and mass density $\varrho_R = 0.001$), which are discretized by 4-node isoparametric displacement-based plain strain elements. The discretization of the outer ring relies on 10x48 elements, for the inner ring 10x40 have been used.
For $t \in [0, 0.5]$, a torque acts on the inner ring in form of a hat function over time. Then, for $t \in (0.5, 2]$, no external loads are acting on the bearing anymore. Fig. 2 shows that for $t \geq 0.5$ the present scheme does indeed conserve the total energy for the frictionless contact problem under consideration.

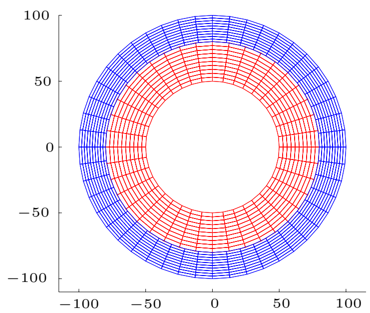

Fig. 1: Discretized bearing

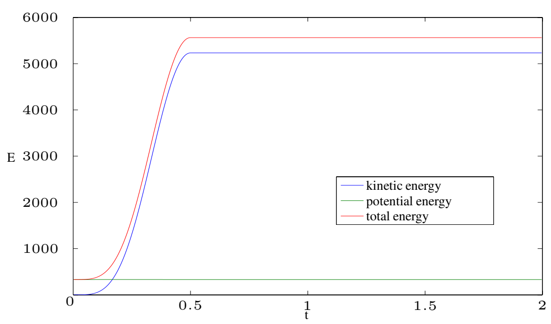

Fig. 2: Energy versus time

Literatur

[1] C. Hesch and P. Betsch, On the energy-momentum conserving integration of large deformation contact problems. *In Proceedings of the ECCOMAS Thematic Conference*, Mailand, 25–28.06.2007
[2] P. Betsch and C. Hesch, Energy-momentum conserving schemes for frictionless dynamic contact problems. Part I: NTS method *IUTAM Bookseries*, Springer-Verlag, 3:77–96, 2007.