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Nonlinear port-Hamiltonian modeling and simulation of rigid body and multibody system dynamics

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Motivation

Objective:

Complex multibody systems (MBS) need robust models that handle large rotations, conserve energy, and remain modular.

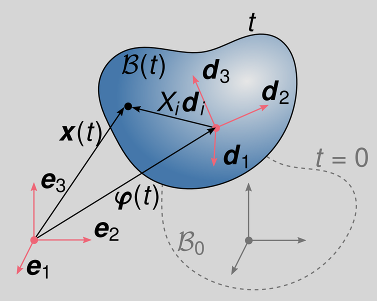
Approach:

- Director formulation + port-Hamiltonian (PH) framework
  - → Singularity-free description of rotations, constant mass matrix, energy consistency, and modular structure.
- Classical kinematic pairs linked to PH interconnections.
- Structure-preserving discretization ensures correct energy and angular momentum balances and avoids drift.

Director formulation [1]

- Kinematics:

$$\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{\varphi}(t) + X_i\,\boldsymbol{d}_i(t)$$



- Rigid body configuration in 12 redundant coordinates:

$$\boldsymbol{q} = (\boldsymbol{\varphi},\ \boldsymbol{d}_1,\ \boldsymbol{d}_2,\ \boldsymbol{d}_3)$$

- Holonomic orthonormality constraints:

$$\boldsymbol{g}_{\mathrm{d}}(\boldsymbol{q}) = \begin{bmatrix} \frac{1}{2}(\boldsymbol{d}_1^\top\boldsymbol{d}_1 - 1) \\ \frac{1}{2}(\boldsymbol{d}_2^\top\boldsymbol{d}_2 - 1) \\ \frac{1}{2}(\boldsymbol{d}_3^\top\boldsymbol{d}_3 - 1) \\ \boldsymbol{d}_1^\top\boldsymbol{d}_2 \\ \boldsymbol{d}_1^\top\boldsymbol{d}_3 \\ \boldsymbol{d}_2^\top\boldsymbol{d}_3 \end{bmatrix} = \boldsymbol{0}$$

PH systems

- State variables: $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{\lambda})$
- PH-DAE formulation, cf. [2]:

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{x})\,\boldsymbol{z}(\boldsymbol{x}) + \boldsymbol{B}(\boldsymbol{x})\,\boldsymbol{u} \quad \text{and} \quad \boldsymbol{E}^\top\boldsymbol{z}(\boldsymbol{x}) = \nabla H(\boldsymbol{x})$$
$$\boldsymbol{y} = \boldsymbol{B}^\top(\boldsymbol{x})\,\boldsymbol{z}(\boldsymbol{x})$$

- $\boldsymbol{J}(\boldsymbol{x}) = -\boldsymbol{J}(\boldsymbol{x})^\top$ encodes (nonlinear) interconnection
- Power balance equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}H = \nabla H(\boldsymbol{x})^\top\dot{\boldsymbol{x}} = \boldsymbol{z}^\top\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{z}^\top\boldsymbol{B}\boldsymbol{u} = \boldsymbol{y}^\top\boldsymbol{u}$$

- For interconnection, decomposition internal and external ports for $\alpha \in \{A, B\}$, cf. [3]:

$$\boldsymbol{B}^\alpha\boldsymbol{u}^\alpha = \begin{bmatrix}\boldsymbol{B}^\alpha_{\mathrm{int}} & \boldsymbol{B}^\alpha_{\mathrm{ext}}\end{bmatrix}\begin{bmatrix}\boldsymbol{u}^\alpha_{\mathrm{int}} \\ \boldsymbol{u}^\alpha_{\mathrm{ext}}\end{bmatrix}$$

Contributions

PH formulation

- PH-DAE formulation for a rigid body:

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{q}} \\ \dot{\boldsymbol{v}} \\ \dot{\boldsymbol{\lambda}}\end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ -\boldsymbol{I} & \boldsymbol{0} & -\nabla\boldsymbol{g}(\boldsymbol{q})^\top \\ \boldsymbol{0} & \nabla\boldsymbol{g}(\boldsymbol{q}) & \boldsymbol{0} \end{bmatrix}\begin{bmatrix}\nabla V(\boldsymbol{q}) \\ \boldsymbol{v} \\ \boldsymbol{\lambda}\end{bmatrix} + \boldsymbol{B}(\boldsymbol{x})\,\boldsymbol{u}$$

- Transformer interconnection for joints:

$$\begin{bmatrix}\boldsymbol{y}^A_{\mathrm{int}} \\ \boldsymbol{u}^A_{\mathrm{int}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{u}^B_{\mathrm{int}} \\ \boldsymbol{y}^B_{\mathrm{int}}\end{bmatrix}$$

- PH-DAE dynamics of two interconnected bodies:

$$\begin{bmatrix} \boldsymbol{E}^A & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E}^B & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{x}}^A \\ \dot{\boldsymbol{x}}^B \\ \dot{\boldsymbol{\lambda}}_{\mathrm{J}}\end{bmatrix} = \begin{bmatrix} \boldsymbol{J}^A & \boldsymbol{0} & \boldsymbol{B}^A_{\mathrm{int}} \\ \boldsymbol{0} & \boldsymbol{J}^B & -\boldsymbol{B}^B_{\mathrm{int}} \\ -(\boldsymbol{B}^A_{\mathrm{int}})^\top & (\boldsymbol{B}^B_{\mathrm{int}})^\top & \boldsymbol{0} \end{bmatrix}\begin{bmatrix}\boldsymbol{z}^A \\ \boldsymbol{z}^B \\ \boldsymbol{\lambda}_{\mathrm{J}}\end{bmatrix} + \boldsymbol{B}_{\mathrm{ext}}\boldsymbol{u}_{\mathrm{ext}}$$

Link to kinematic pairs

- PH interconnection contains exact same information as kinematic pair for ideal constraints
- Analogy provides rigorous link between PHS from systems theory and classical multibody dynamics
- Definition velocity-related port matrix $\tilde{\boldsymbol{B}}^\alpha(\boldsymbol{x})$:

$$\boldsymbol{B}^\alpha(\boldsymbol{x})\,\boldsymbol{u} = \begin{bmatrix}\boldsymbol{0} \\ \tilde{\boldsymbol{B}}^\alpha(\boldsymbol{x}) \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{u}$$

- Key observation:

$$\begin{bmatrix}-(\tilde{\boldsymbol{B}}^A_{\mathrm{int}})^\top & (\tilde{\boldsymbol{B}}^B_{\mathrm{int}})^\top\end{bmatrix}\begin{bmatrix}\boldsymbol{v}^A \\ \boldsymbol{v}^B\end{bmatrix} = \nabla\boldsymbol{g}_{\mathrm{J}}(\boldsymbol{q})\,\boldsymbol{v}$$

Time integration

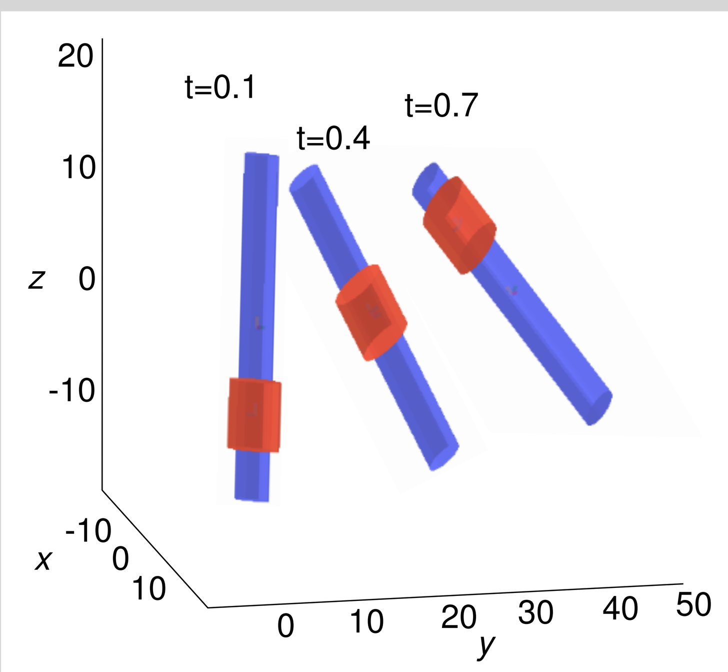
- One-step implicit midpoint rule with $\boldsymbol{x}^{n+\frac{1}{2}} = \frac{1}{2}(\boldsymbol{x}^{n+1} + \boldsymbol{x}^n)$
  - 2nd-order accurate
  - Consistent energy and angular momentum balances
  - Position level constraints satisfied
- Time-discrete update equations:

$$\boldsymbol{E}(\boldsymbol{x}^{n+1} - \boldsymbol{x}^n) = h\,\boldsymbol{J}(\boldsymbol{x}^{n+\frac{1}{2}})\,\boldsymbol{z}^{n+\frac{1}{2}} + h\,\boldsymbol{B}(\boldsymbol{x}^{n+\frac{1}{2}})\,\boldsymbol{u}^{n+\frac{1}{2}}$$
$$\boldsymbol{E}^\top\boldsymbol{z}^{n+\frac{1}{2}} = \nabla H(\boldsymbol{x}^{n+\frac{1}{2}})$$
$$\boldsymbol{y}^{n+\frac{1}{2}} = \boldsymbol{B}(\boldsymbol{x}^{n+\frac{1}{2}})^\top\boldsymbol{z}^{n+\frac{1}{2}}$$

- Consistent discrete-time power balance:

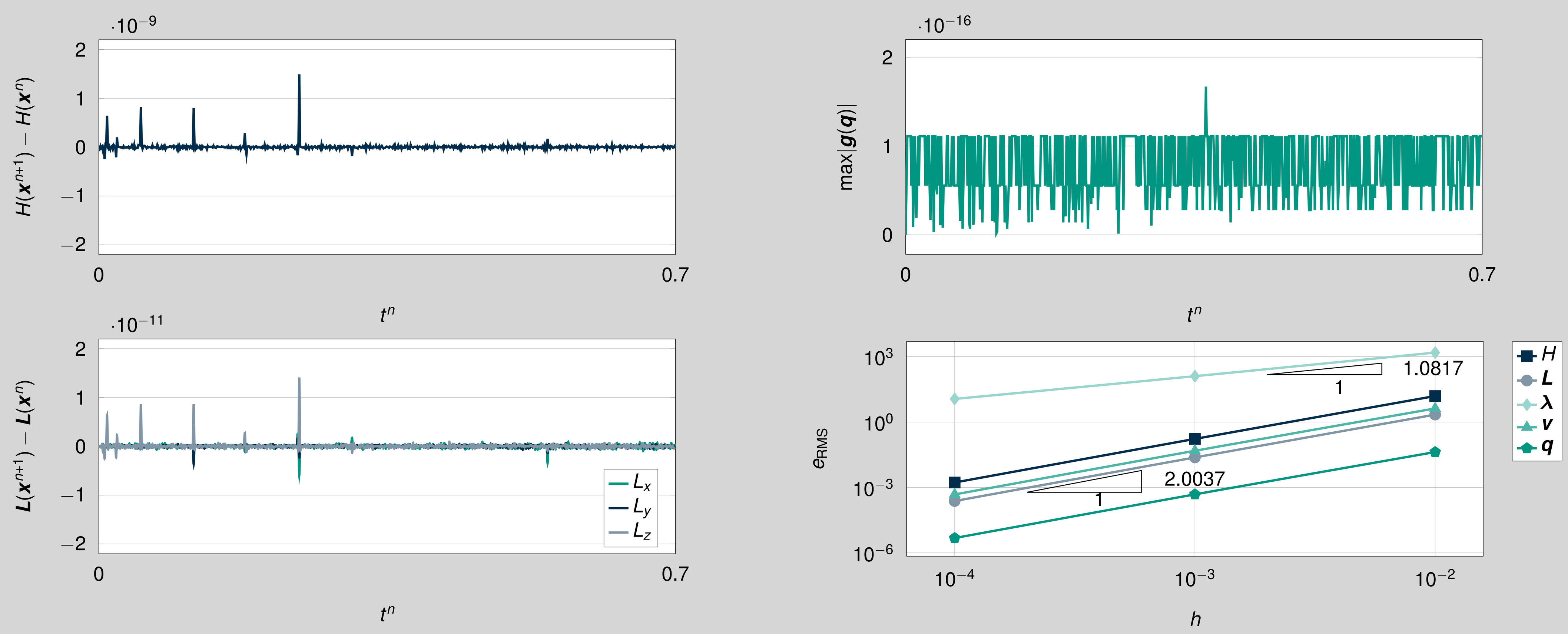
$$H(\boldsymbol{x}^{n+1}) - H(\boldsymbol{x}^n) = h\,(\boldsymbol{y}^{n+\frac{1}{2}})^\top\boldsymbol{u}^{n+\frac{1}{2}}$$

Numerical example [4]



Flying cylindrical pair

Properties



References

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